A FIBONACCI ARRAY

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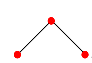
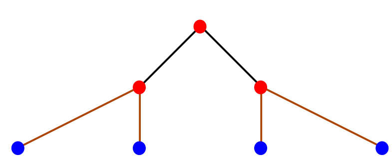
1. Introduction

We will define a certain numerical array, which we call the *Fibonacci array* $\mathfrak{F}$, and will state some properties of this array related to Fibonacci numbers and the golden mean. Proofs are omitted; for further details see the reference at the end of this article.

Define a diagram as follows. At the top there is a single vertex (or point or node), denoted T (for "top"). Now continue recursively using the following rules:

- (P1) Each vertex is connected to exactly two vertices in the row below.
- (P2) The diagram is planar, i.e., edges cannot cross.
- (P3) Given a vertex t and the two adjacent vertices u, v to t in the row below, complete this figure to a hexagon by adding a vertex u' below and adjacent to u, a vertex v' below and adjacent to v, and a vertex w below and adjacent to both u' and v'.

Thus the first step is to add two vertices below T: . We cannot add a vertex below both of the two bottom vertices, because we must complete to a hexagon, not a quadrilateral. Since the two bottom vertices must each be adjacent to two vertices below, at the next step we get



Now we add a vertex adjacent to the two middle vertices on the bottom row in order to complete to a hexagon:

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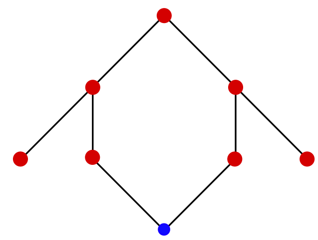
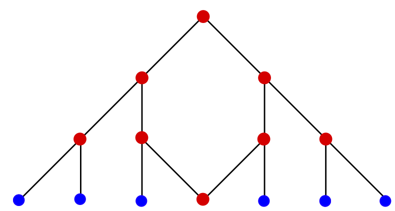
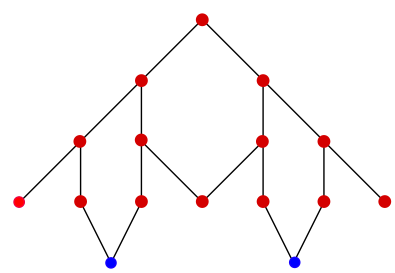


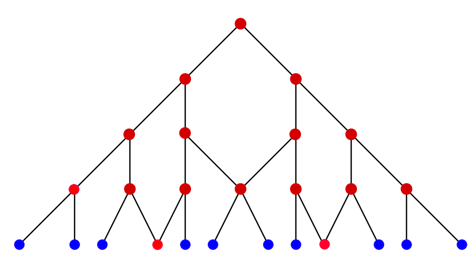
Add remaining vertices on bottom row so that rule (P1) is satisfied:



Complete the two hexagons:



Add remaining vertices on bottom row:



Continuing in this manner produces a diagram consisting of infinitely many levels. We denote this diagram by $\mathcal{D}$. The top element T is defined to be at level 0. The two vertices immediately below T are at level one, etc. The number of vertices at the levels $0, 1, 2, \dots$ is $1, 2, 4, 7, 12, 20, 33, 54, \dots$. In fact, the number of vertices at level n is $F_{n+3}-1$, where F_i denotes a Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_{i+1} = F_i + F_{i-1}$ for $i \geq 2$). This gives the first glimpse of the connection of our diagram with Fibonacci numbers.

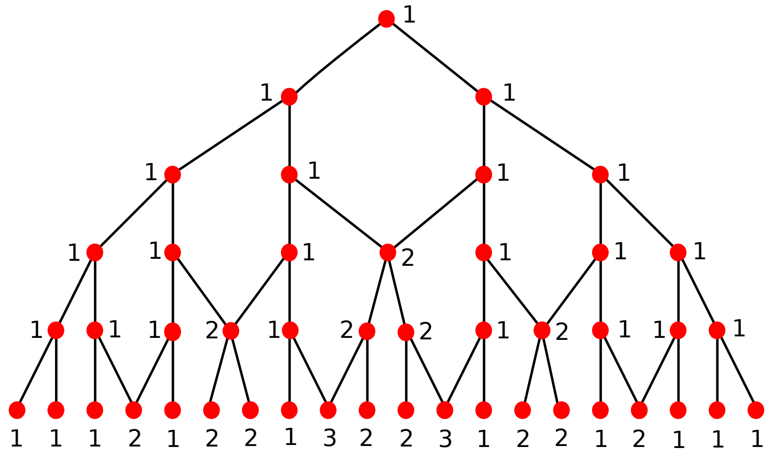


FIGURE 1. The Fibonacci array $\mathfrak{F}$

The next step is to attach a positive integer (a label) to each vertex of $\mathcal{D}$ by the following recursive procedure. The top element T is labelled 1. Once we have labelled all the vertices at level n, label a vertex v at level $n+1$ by the sum of the labels of the elements on level n that are adjacent to v. This procedure is analogous to the usual recursive definition of Pascal's triangle[1]. A nonrecursive description of the label of a vertex v is that the label is equal to the number of paths from T to v (along the edges of the diagram $\mathcal{D}$). We denote the resulting labelled diagram by $\mathfrak{F}$, called the *Fibonacci array*. Figure 1 shows the levels 0 to 5 of $\mathfrak{F}$.

2. The numbers $\left\langle {n \atop k} \right\rangle$

What are the numbers appearing in $\mathfrak{F}$? Let $\left\langle {n \atop k} \right\rangle$ denote the kth number on level n of $\mathfrak{F}$, beginning with $k=0$. Thus for instance from Figure 1 we see that

$$\left\langle {5 \atop 0} \right\rangle = \left\langle {5 \atop 1} \right\rangle = \left\langle {5 \atop 2} \right\rangle = 1,\ \left\langle {5 \atop 3} \right\rangle = 2, \left\langle {5 \atop 4} \right\rangle = 1, \dots.$$

The numbers $\left\langle {n \atop k} \right\rangle$ may be regarded as "Fibonacci analogues" of the binomial coefficients $\binom{n}{k}$. The binomial coefficients satisfy the binomial theorem

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = (1+x)^n. \tag{2.1}$$

[1]In fact, if we modify the rule (P3) by saying that we complete a vertex and the two adjacent vertices u, v to a quadrilateral rather than a hexagon and use the same labeling rule, then we obtain Pascal's triangle.

The numbers $\left\langle {n \atop k} \right\rangle$ satisfy

$$\left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle x + \left\langle {n \atop 2} \right\rangle x^2 + \cdots + \left\langle {n \atop F_{n+3}-2} \right\rangle x^{F_{n+3}-2}$$

$$= (1+x^{F_2})(1+x^{F_3})\cdots(1+x^{F_{n+1}}), \tag{2.2}$$

a "Fibonacci analogue" of the binomial theorem. For instance,

$$(1+x)(1+x^2)(1+x^3)(1+x^5)$$

$$= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11},$$

so the labels on the fourth level of $\mathfrak{F}$ are $(1,1,1,2,1,2,2,1,2,1,1,1)$.

3. Sums of powers of $\left\langle {n \atop k} \right\rangle$

In Pascal's triangle the sum of the numbers on level n is 2^n. In symbols,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

This formula follows from the fact that every number in Pascal's triangle is used twice in forming the next row. Alternatively, we can set $x = 1$ in the binomial theorem (2.1). Exactly the same reasoning applies to the Fibonacci array. Each number on some row is used twice in forming the next row, essentially a restatement of property (P1). Alternatively, we can set $x = 1$ in equation (2.2), so we get

$$\left\langle {n \atop 0} \right\rangle + \left\langle {n \atop 1} \right\rangle + \cdots + \left\langle {n \atop F_{n+3}-2} \right\rangle = 2^n. \tag{3.1}$$

The situation becomes more interesting when we consider powers $\left\langle {n \atop k} \right\rangle^r$ of the entries. The main result is the following. Let r be a positive integer, and set

$$v_r(n) = \left\langle {n \atop 0} \right\rangle^r + \left\langle {n \atop 1} \right\rangle^r + \cdots + \left\langle {n \atop n} \right\rangle^r.$$

Thus $v_1(n) = 2^n$, a restatement of equation (3.1). In general, $v_r(n)$ satisfies a linear recurrence with constant coefficients, i.e., there are integers $c_1, ..., c_k$ (which depend on r, as does k) such that

$$v_r(n) = c_1 v_r(n-1) + c_2 v_r(n-2) + \cdots + c_k v_r(n-k)$$

for all $n \geq k$. For instance,

$$\begin{aligned}
v_2(n) &= 2v_2(n-1) + 2v_2(n-2) - 2v_2(n-3) \\
v_3(n) &= 2v_3(n-1) + 4v_3(n-2) - 2v_3(n-3) \\
v_4(n) &= 2v_4(n-1) + 7v_4(n-2) + 2v_4(n-4) - 2v_4(n-5) \\
v_5(n) &= 2v_5(n-1) + 11v_5(n-2) + 8v_5(n-3) \\
&\quad +20v_5(n-4) - 10v_5(n-5).
\end{aligned}$$

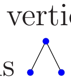
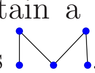
Nothing like this is true for the ordinary binomial coefficients $\binom{n}{k}$.

Note (for readers with sufficient mathematical background). Define the power series $V_r(x) = \sum_{n\geq 0} v_r(n)x^n$. Since $v_r(n)$ satisfies a linear recurrence with constant coefficients, $V_r(x)$ is a rational function. For $1 \leq r \leq 6$ it is given by

$$\begin{aligned}
V_1(x) &= \frac{1}{1-2x} \\
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3} \\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5} \\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}.
\end{aligned}$$

Note that the numerator of $V_r(x)$ is the "even part" of the denominator. It was proved by Ilya Bogdanov that this fact continues to hold for any r (MathOverflow 457900).

4. Two consecutive levels

We now turn to a completely different aspect of $\mathfrak{F}$: the structure of two consecutive levels. Consider for instance levels four and five, shown as blue vertices in Figure 2. We obtain a sequence of three-vertex diagrams  and five-vertex diagrams . Thus we can represent the structure of two consecutive levels as a sequence of 3's and 5's. For instance, rows 4 and 5 correspond to the sequence $(3,5,3,5,5,3,5,3)$. In general, the number of terms in the sequence corresponding to rows n and $n+1$ is F_{n+2}.

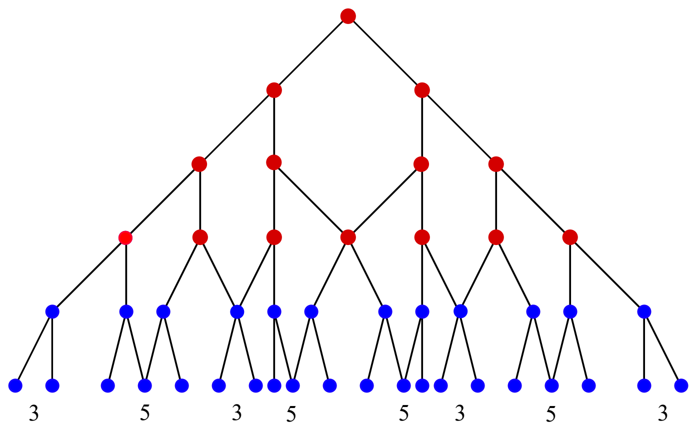


FIGURE 2. Levels four and five of $\mathfrak{F}$

How can we describe the sequence corresponding to levels n and $n+1$? It is palindromic (reads the same backwards as forwards), so we only have to describe the first half. The result is that the kth term (beginning with $k=1$) is given by

$$1 + 2\lfloor k\phi \rfloor - 2\lfloor (k-1)\phi \rfloor, \tag{4.1}$$

where $\phi = (1+\sqrt{5})/2$, the *golden mean*. As usual, $\lfloor x \rfloor$ denotes the greatest integer $m \leq x$.

The numbers in equation (4.1), beginning with $k=1$, are

$$\gamma = (3,5,3,5,5,3,5,3,5,5,3,5,5,3,5,3,5,5,\dots). \tag{4.2}$$

The first four terms are $3,5,3,5$, agreeing with the description of the first half of levels 4 and 5.

The sequence (4.2) has several other descriptions.

- If we remove the first term, then the remaining sequence (5,3,5,5,3,5,...) is characterized by invariance under $3 \to 5$ and $5 \to 53$ (the *Fibonacci word* in the letters 3,5).
- We have $\gamma = 3z_1z_2z_3\cdots$ (concatenation of words), where $z_1 = 5$, $z_2 = 35$, and $z_k = z_{k-2}z_{k-1}$ for $k \geq 3$:

$$(3)\, 5 \cdot 35 \cdot 535 \cdot 35535 \cdot 53535535 \cdots .$$

- If we replace 3 by 1 and 5 by 2 in γ, then we obtain the sequence that records the number of 5's between consecutive 3's in γ:

$$3\underbrace{5}_{1}3\underbrace{55}_{2}3\underbrace{5}_{1}3\underbrace{55}_{2}3\underbrace{55}_{2}3\underbrace{5}_{1}3\underbrace{55}_{2}3\cdots .$$

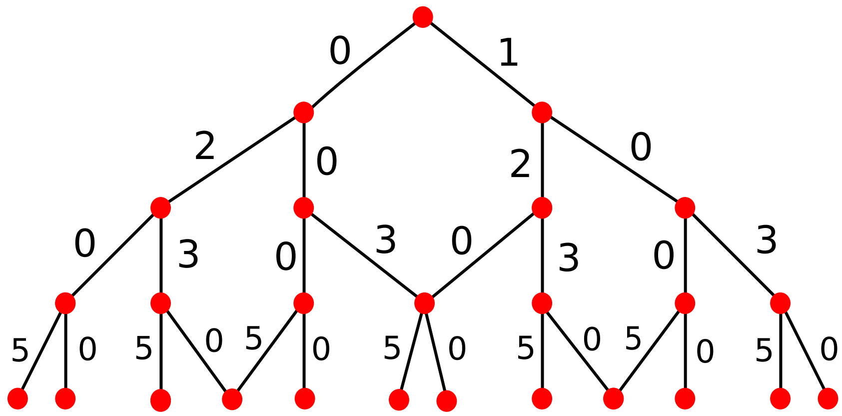


FIGURE 3. An edge labeling of $\mathfrak{D}$

5. An edge labelling

Label the edges of $\mathfrak{D}$ as follows. The edges between levels $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right. The edges between levels $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right. Figure 3 shows the first four levels of this labeling.

If t is a vertex in $\mathfrak{D}$, then the sum $\sigma(t)$ of the edge labels on any path from t to the top depends only on t, not on the choice of path. Figure 4 shows these sums for the points at level four. At level n we obtain the integers from 0 to $F_{n+2}-2$ once each. As we go down a path from the top to level n, there are two choices for each step. These choices correspond exactly to expanding the product (2.2). For each of the n factors there are two choices: choose the constant term 1 or the monomial $x^{F_{i+1}}$.

Moreover, if i appears to the left of j at level n, then i appears to the left of j at all subsequent levels. Thus we can define a linear ordering, denoted $\prec$, on the nonnegative integers by letting $i \prec j$ if i appears to the left of j at some level n (and thus at all subsequent levels). Figure 4 shows that

$$7 \prec 2 \prec 10 \prec 5 \prec 0 \prec 8 \prec 3 \prec 11 \prec 6 \prec 1 \prec 9 \prec 4.$$

The order $\prec$ on the nonnegative integers is *dense*, meaning that whenever $i \prec k$, there is some (hence infinitely many) j satisfying $i \prec j \prec k$. The description of this order is based on *Zeckendorf's theorem*, which says that every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2 . The description of the order $\prec$ is a little

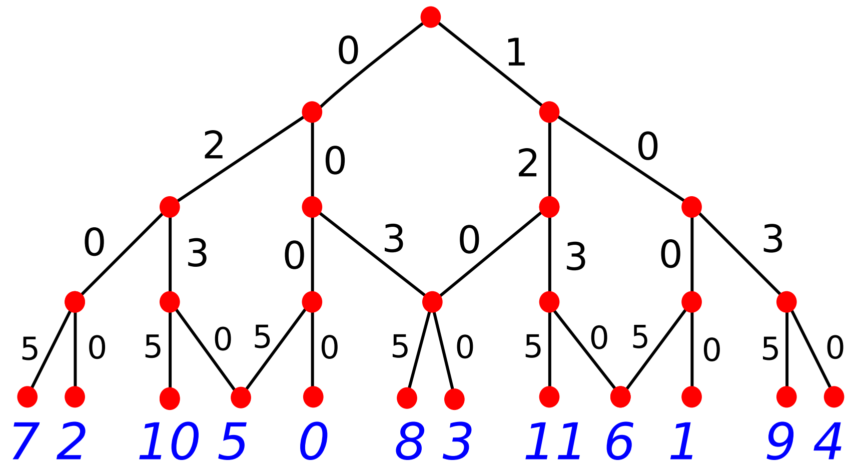


FIGURE 4. An ordering of the integers from 0 to 11

too complicated to describe here, but to give the flavor we give the condition for $n \succ 0$. Namely, let $n = F_{j_1} + \cdots + F_{j_s}$ be the Zeckendorf representation of $n > 0$, where $j_1 < \cdots < j_s$. Then $n \prec 0$ if j_1 is odd, while $n \succ 0$ if j_1 is even. For instance, $45 = 3 + 8 + 34 = F_4 + F_6 + F_9$. Since the first index (subscript) 4 is even, we have $45 \succ 0$.

REFERENCE. R. Stanley, Theorems and conjectures on some rational generating functions, *Europ. J. Math.*, to appear; `arXiv:2101.02131`.

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