Hints for Designing Puzzles

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From the perspective of a designer of various mathematical puzzles, including mechanical puzzles, this article carefully selects puzzles deemed suitable for discussion in this context, from those I have designed so far. It introduces these puzzles and shares the background of their creation. (This article is a translation, with minor revisions made to the original Japanese text, of a contribution originally published in the March 2023 issue (Volume 68, Issue 3) of *Operations Research*, a journal of the Operations Research Society of Japan. It was translated by ChatGPT and Iwahiro.)

1. Introduction

Let me begin with a brief introduction about myself, essential for understanding the purpose of this paper. I am somewhat known in the world of mechanical puzzle design. My recognition is partly due to my reasonably successful track record in the International Puzzle Design Competition (officially known as the Nob Yoshigahara Puzzle Design Competition) [1], including two top awards [2, 3]. Though I produce work sparingly, I am recognized by puzzle collectors worldwide for my extremely simple and unique designs rooted in a mathematical background. Therefore, I sometimes refer to myself as a "puzzle designer," though I am more of a mathematical puzzle researcher. I have written and translated numerous books and articles on mathematical puzzles and recreational mathematics. For several years now, I have also had the opportunity to pose a problem for the "Seeking Elegant Solutions" column in the *Mathematics Seminar* (数学セミナー) magazine, so Japanese readers of that column might be familiar with my name (the latest article at the time of writing this G4G15 paper is [4]. A book [5] collecting masterpieces from the same column also includes a problem I have posed).

While I don't like to refer to my activity of creating mathematical puzzles, including mechanical puzzles, as merely a "hobby," it's true that my main profession is different – I am an expert in actuarial science. Moreover, I don't create enough puzzles to consider it a full-time profession. However, even so, I have devised several puzzles that I consider masterpieces according to my own values, and I intend to continue doing so. A common feature of these puzzles is a certain kind of mathematical beauty, and I intend to create only those puzzles that embody this beauty. Occasionally, there are people who share a similar aesthetic sense to mine and who highly praise my puzzles. I understand that the invitation to contribute to this issue of *Operations Research*, which is a special feature on "The Conception of Puzzles," is a result of such connections.

In this article, I humbly present some stories about puzzle design from my perspective as a sparingly productive puzzle creator who is dedicated to some type of mathematical beauty. Through the following descriptions, I hope to convey some of the delights of devising new creations, and it would be extremely gratifying if this could serve as a reference or inspiration in any way.

2. Triangular Jam

2.1 Introduction of the puzzle

One of the mechanical puzzles I have designed is "Triangular Jam" (see Figure 1. While this puzzle has been commercialized several times under various names, the version shown in the figure is the original one I personally produced).

Figure 1: Triangular Jam

It would be quite challenging for readers to make

“Triangular Jam” themselves, so I will introduce a puzzle that is essentially almost the same. Even with this alternative puzzle, it's better to physically make and work with it, but I believe the enjoyment can still be conveyed reasonably well without making one.

The puzzle consists of a square frame and four identically sized equilateral triangular pieces. One of these four triangles should be distinguishable from the others, for example by painting it red. If the length of one side of the square frame (inner dimension) is set to 1, then the length of one side of the equilateral triangles is $1/\sqrt{3}$. The starting arrangement is as shown on the left in Figure 2, and the goal is on the right in the same figure (in the original puzzle, the goal is to remove the red piece, which is a thin board that can be extracted through a tunnel on the bottom edge of the frame).

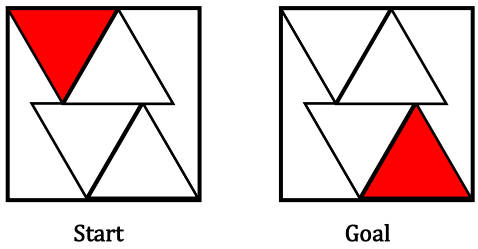


Figure 2: A variant of Triangular Jam

As a puzzle, the objective is of course to move from the start to the goal, but during this process, the pieces must not be lifted at all. That is, the goal must be reached only by sliding the pieces, and in this respect, this puzzle is a “sliding puzzle.” Additionally, of course, it's NOT a joke puzzle where one could simply rotate the frame or change their viewing angle without moving anything and claim “Look, I've solved it.”

This puzzle is interesting because, from the solver's initial perspective, it's surprising that reaching the goal is possible given the seemingly too crowded arrangement of triangular pieces. The key move for solving it, which I won't disclose here to maintain the reader's enjoyment, can be quite inspiring for some. When discovered, it often brings a sense of satisfaction or delight to solvers.

In terms of dimensions, if the equilateral triangles are made even slightly larger (relative to the frame), moving the top-left triangle to the bottom-right corner would become impossible. Given that this limitation is almost self-evident, the specific design of the puzzle can be seen as a kind of optimization problem. To demonstrate that this length is optimal in the design, one simply needs to solve the puzzle.

2.2 The Origine of the Idea

How did I come up with such a puzzle, though?

At that time, I was aiming to create a “revolutionary” puzzle in the sense that it wouldn’t fit neatly into traditional puzzle classifications. Triangular Jam, as mentioned earlier, falls into the category of sliding puzzles. Given this, it might seem not at all revolutionary. However, for puzzle enthusiasts, this is not the case.

Puzzle enthusiasts commonly use two classification methods for mechanical puzzles: the Slocum classification and the Dalgety-Hordern classification. In both, sliding puzzles are categorized as a sub-type of Sequential Movement Puzzles (SMP). SMPs are puzzles where each step in the solving process is clearly distinguishable from others, and the choices at each step are limited to a few simple options. This includes puzzles like the 15 Puzzle, the Tower of Hanoi, and the Rubik's Cube. Therefore, in principle, they are puzzles that can be solved with simple search algorithms (although there are cases where they become NP-problems, making them challenging for computers, or finding the shortest solution is difficult).

Then, I set my sights on creating a sliding puzzle that was not an SMP. This idea stands even apart from classifications only known within the puzzle community. In essence, since all traditional sliding puzzles could, in principle, be solved with search algorithms, my aim was to design something “revolutionary” by creating a puzzle that deviated from this norm.

The rest was simply a search for something that used as few types and as few pieces as possible, while still being mathematically simple and aesthetically pleasing, yet not too easy to solve, and possessing some element of surprise. In fact, using just one type of equilateral triangle was my first option, and four pieces seemed almost the bare minimum for a sliding puzzle, making the conception of this puzzle quite natural. Therefore, after I thought of it, it seemed so straightforward that it made me wonder why such a puzzle hadn't existed before, and it felt like it wouldn't have been surprising if it had been invented long ago whether its inventor is known or unknown.

In any case, this puzzle was a logical creation born out of the ambition to devise something revolutionary. Furthermore, the sentiment that “it would not be surprising if it had been invented long ago” is an important criterion (though not necessarily essential) for my satisfaction with a puzzle I have designed.

3. Rectangular Jam

3.1 Introduction of the Puzzle

Believing that it would be interesting to have a genre of sliding puzzles with features similar to Triangular Jam, I devised several puzzles of this type. Among these, the one I consider to be my greatest masterpiece is “Rectangular Jam” (see Figure 3. While there are various versions of this puzzle, the version shown is the original one I personally produced, marked with a branding iron. Depicted nearly in its goal state).

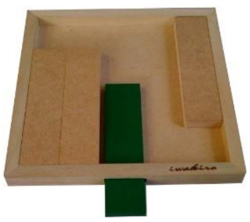
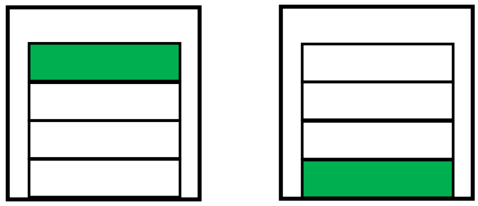


Figure 3: Rectangular Jam

This one would also be difficult to make oneself, so I will introduce an outline of a variant. I say “outline” because I will not provide the precise dimensions of the pieces, in order to preserve the reader's enjoyment (as will be mentioned later).

The puzzle consists of a single square frame and four identically sized rectangular pieces, each with an aspect ratio of 1:4. One of these four rectangles should be distinguishable from the others, for example by painting it green. The starting arrangement is as shown on the left in Figure 4, and the goal is on the right in the same figure. The remaining rules are exactly the same as those for the variant of Triangular Jam (the differences from the original puzzle are also similar to those in the case of Triangular Jam).



Start **Goal**

Figure 4: A variant of Rectangular Jam

3.2 The Origin of the Idea

The process of conceiving this puzzle was quite different from that of Triangular Jam. I say this because, initially, I did not expect that a puzzle with such a simple arrangement could be challenging. Hence, it was not conceived in a logical way, as was the case with Triangular Jam.

I came up with this puzzle when I was fiddling with rectangular pieces and noticed an interesting movement. This notice led me to think, "What if, for 1:4 rectangles, this movement is the optimal among all possible movements? Then, a puzzle in the same genre as Triangular Jam could be created with a remarkably beautiful arrangement?" Even with this idea, I experimented skeptically, thinking, "It can’t be that straightforward." But to my surprise, it turned out that this movement was indeed the optimal one. The moment I realized this was truly a joyous one.

I believe this puzzle was born as a result of combining my constant pursuit of interesting movements that could potentially become a puzzle, with the idea of creating a puzzle in this genre always lingering in the back of my mind.

3.3 Design Method and Unsolved Problem

I am often asked about how I designed the dimensions of this puzzle. The question varies in meaning depending on who asks it, but it can be interpreted as asking how I solved the following mathematical problem:

Problem: Find the largest possible dimensions of the rectangle with a 1:4 aspect ratio used for four pieces, such that they can reach the goal from the start position as shown in Figure 4, by moving only through sliding.

This problem can be said to have been solved, of course, but since I have never written down a precise mathematical proof, it remains an "unsolved problem" in that respect. I would definitely encourage you to give it a try.

Solving this puzzle itself involves a key movement, and determining the largest dimensions for which this movement works using numerical calculations with a computer is not too difficult (for most readers of this article). Of course, to actually design it, one must first identify the key movement. Finding this movement without physically manipulating the actual puzzle can be quite challenging.

3.4 Solvable by Computer?

There is another problem that I think might interest the readers of this article. This issue is related to the purpose for which Triangular Jam and Rectangular Jam were designed, so I will introduce it here. The problem is as follows:

Problem: Although the number of possible states during the process of solving Rectangular Jam is infinite, can Rectangular Jam be solved by a computer using a search algorithm in a broad sense?

The intention of the creator of Triangular Jam and Rectangular Jam was to devise puzzles that could not be solved by search algorithms. Therefore, solving this problem is, in a sense, a true challenge to these puzzles. If you are interested, I would like to encourage you to take up the challenge. There is also some prior research [6].

4. Sliding Coin Puzzle: "Four Coins"

4.1 Common Rules

Let's take a brief detour from mechanical puzzles and introduce a single problem (not counting the example problem for explaining rules) from a category known as "sliding coin puzzles." These puzzles use only ordinary coins instead of specialized tools. The problem is quite simple, but before delving into it specifically, it's necessary to first understand the common rules of sliding coin puzzles.

The following example problem is a classic sliding coin puzzle.

Example Problem: Arrange six coins as shown on the left in Figure 5. Starting from this arrangement, move one coin per step and reach the configuration shown on the right in Figure 5 in three steps.

Figure 5: A sliding coin puzzle

These sliding coin puzzles follow these rules:

Rule 1: Each step involves moving only one coin, ensuring no other coins are moved, either directly or indirectly.

Rule 2: Lift no coins; only slide them.

Rule 3: Move each coin to a position where it becomes adjacent to at least two other coins, thus determining its placement.

Rule 4: The goal diagram represents the coins' relative positions; their exact location and orientation may vary.

4.2 Introduction of the Puzzle

The problem I devised, which I consider a masterpiece, is as follows:

Problem: Arrange four coins as shown on the left in Figure 6. White represents heads-up coins, and black represents tails-up coins. Starting from this arrangement, move one coin per step and reach the configuration shown on the right in Figure 6 in three steps.

Figure 6: "Four Coins"

Since it can be solved in just three steps, it shouldn't be too difficult. However, some people feel that it is unsolvable. Additionally, many who have solved it once often forget how they did it after a while. Solving it in four steps is not very easy either, so I suggest starting by trying to solve it in four steps.

4.3 The Origin of the Idea

Is it an exaggeration to say that it took decades to develop this puzzle?

The basic idea for this sliding coin puzzle has been on my mind since my teens. I didn't come up with it myself; I heard it from someone, though I don't remember who. Although I always thought the idea was interesting, directly turning it into a puzzle didn’t produce an elegant question and was not satisfying. After considering it for many years, I suddenly realized that using the heads and tails of coins was the ideal way to implement this idea as a puzzle, leading to the creation of the problem mentioned above.

Therefore, I believe the birth of this puzzle resulted from long contemplating an idea I always found interesting. I still have many other ideas that I've been considering for a long time, but it's challenging to turn them into actual puzzles. Later, I will introduce another example where a long-contemplated idea was successfully realized.

4.4 A Hint

Providing hints might seem superfluous, yet I understand that for some who couldn’t solve the puzzle, it might seem like a joke. To address this concern, I will offer a hint below, though I still won't reveal the solution.

I wrote earlier, “I suggest starting by trying to solve it in four steps.” However, if you can solve in four steps and your solution is relatively straightforward, trying to reduce it to one fewer step might not help you solve the puzzle. Of course, since it's only three steps, exploring all possibilities should quickly reveal the solution (considering symmetries, the number of cases isn't so large). But, if you don't pay enough attention to Rule 4, you might almost reach the goal but fail to realize it. That is, you must be fully aware that the overall goal configuration can be rotated compared to what is shown on the right in Figure 6.

5. Probability Puzzle: “Coin Toss Challenge”

5.1 Introduction of the Puzzle

The area of mathematical puzzles in which I specialize the most is probability puzzles. I have introduced many problems in various articles and books. I have even authored a book [7] solely dedicated to probability puzzles. Here, I will introduce just one problem.

This problem, which I created relatively recently, is one I'm particularly fond of. I introduce it partly because I thought it was perfectly suited for the readers of *Operations Research*. I recommend approaching it not as a puzzle at first, but as if you've encountered essentially the same problem during some research process.

Problem: Two players each have five fair coins. Since there are ten sides in total for the five coins, each player writes numbers from 1 to 10 (using each number only once) on them as they like. Once they're ready, both players toss their five coins, and the player with the larger product wins (if the products are the same, they keep tossing until a winner is determined). How should the numbers be written on the coins to maximize the chances of winning? If both use the best strategy, the game naturally results in a fifty-fifty chance of winning, but is there a way to ensure a winning probability of at least fifty percent regardless of the opponent's strategy? If such a way exists, what is it? If there are multiple equivalent ways, list them all.

5.2 Enhancing Problem Experience

This problem may seem like a simple exercise in finding the optimal strategy. Indeed, it is likely a good practice problem. Since each player has five coins, the number of cases to consider is considerable (though too many for a human to handle easily), but it's not too daunting with the use of a computer. Or, since the goal is just to find the optimal solution, you might not want to start with a full search immediately. Instead, it might be wise to think of a few seemingly good strategies and do some simulations first. In any case, I feel this problem is particularly well-suited as a practice problem for students studying in fields related to operations research.

Additionally, while it may not guarantee the optimal strategy, as a first step to get a rough idea, you might try finding a strategy that maximizes the expected value of the product. This in itself is an interesting mathematical exercise and, in fact, leads to a solution with a beautifully symmetrical result.

I believe that this problem can be enjoyed as various kinds of challenges, as described above. Also, no matter how one approaches it, upon finally arriving at the answer, I believe it will be quite surprising for many people. On the other hand, and perhaps "of course," since this is a puzzle, there is also a quick solution without the use of a computer. Unlike the other puzzles introduced in this article, the solution to this puzzle will be provided within this article. Therefore, I recommend thoroughly enjoying it before looking at the answer.

5.3 The Origin of the Idea

I came up with this problem after hearing the following problem from someone:

Problem I Heard: Prepare three standard six-sided dice (without any bias) and one twenty-sided die (also unbiased), numbered from 1 to 20. When rolling these dice, compare the sum of the numbers on the three six-sided dice with the number on the twenty-sided die. The higher number wins. Which has the advantage in this game?

For this problem (I will provide the answer immediately in this paragraph, so those who want to solve it themselves should be cautious), I later came up with an elementary and elegant solution myself. However, when I first heard it, I quickly realized the answer using a rather abstract and general principle. The answer is that “neither has an advantage; it is fair.” The principle I used at that time was that “if there are two symmetrical probability distributions with the same mean, then for any two independent random variables X and Y following their respective distributions, $P(X < Y)$ equals $P(X > Y)$.

While I quickly realized the answer, I thought it better to avoid using advanced principles when explaining it to others. So, I pondered over the mathematical aspects of this problem and discovered several intriguing points. I endeavored to encapsulate this interest in the form of a puzzle, aiming to make it engaging in various ways, as described in 5.2. The main innovation in posing this puzzle was to maintain its simplicity while ensuring the answer was not easily guessable, and the inherent symmetry was not overtly obvious.

When I encountered an interesting idea and tried to turn it into a puzzle, it worked out quite well. In that respect, I think the creation of this puzzle was similar to the development of Rectangular Jam.

5.4 Answer and Solution

I believe the charm of this problem won’t be fully conveyed unless the solution is also presented, so I’ll describe it below.

Before the solution, let me first reveal the answer, which might be surprising if you haven’t yet arrived at it yourself: “you can write the numbers any way you like, as long as the rules are followed.” In other words, “no matter how the numbers are written, there is no advantage or disadvantage.” The phrase “you can write any way you like” doesn't mean that you need a random selection like choosing a hand in rock-paper-scissors. In fact, even in a situation where you are allowed to write your numbers after seeing your opponent’s, it’s still impossible to write them in a way that creates any advantage or disadvantage.

There is a straightforward solution to understand why this is the answer. No matter how the numbers are written, as long as the rules are followed, the product of all numbers on both sides of the coins is the same. Therefore, in any pair of coin tosses, if one player wins in a game viewed from the top, they are losing when viewed from the bottom. Since the probability of winning when viewed from the top is naturally the same as from the bottom (due to symmetry), no matter how the numbers are written, the probability of winning equals the probability of losing, which is 50% (note that if the products are the same, the game continues until a winner is determined).

6. Card in the Bag

6.1 Introduction of the Puzzle

As the last puzzle to be introduced in this article, I present another mechanical puzzle named "Card in the Bag." Though it is a mechanical puzzle (meaning it requires specialized tools), this puzzle, as explained below, can be easily made by anyone. I highly encourage you to make it yourself and then give it a try.

Prepare a credit card or a card of the same size, and a plastic bag with internal dimensions of 10 cm in width and 5 cm in length. That's all you need for the setup (see Figure 7). When you try to insert the card directly into the bag, it will protrude slightly. The goal of the puzzle is to completely enclose the card inside the bag without deforming the card in any way (including during the process of solving the puzzle). Of course, cutting or tearing the bag is also not allowed.

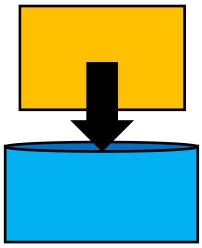


Figure 7: Card in the Bag

The fascination of this puzzle truly comes to light only when one actually tries it. Some people may think they have "solved it" just by thinking about it, but often they haven't figured out how to implement their solution in reality. If you think you've solved it, it's best to either try it with an actual card and bag or, at the very least, carefully verify the exact steps of your proposed solution.

6.2 Tips for Properly Making It Yourself

Since I strongly encourage you to create and try this puzzle yourself, I will provide some tips and cautions for properly making it.

Various materials are used for poly bags, so avoid those that tear easily or irreversibly stretch with little force. If you have a poly bag exactly 10 cm wide, cutting it to a length of 5 cm will suffice. However, be cautious: even if a product is marked as 10 cm, the internal width might be shorter, so ensure that the internal width is indeed 10 cm. If it's slightly larger, up to about 1 mm more is acceptable.

The length is mentioned as 5 cm, but this is primarily for the simplicity of a round number and ease of remembering. Actually, it can be about 2 mm shorter than this. Therefore, there's no need to be overly precise with the length; as long as it's close to 5 cm and not longer, it should be fine.

6.3 The Origin of the Idea

For this puzzle as well, I will not provide the solution in this article. However, I'd like to caution readers regarding the timing of reading the following section, as it significantly hints at the solution within the story of its origin.

I have always found a certain special property of some tetrahedra fascinating since my teens. It was only relatively recently that I tried to articulate this property in words. While a technical term may already exist in some field unfamiliar with me, I have informally called it "Untaperedness (寸胴性)."

Untaperedness is defined as follows:

Definition: A solid is said to be untapered if there exists a plane such that, when the solid is sliced by any plane parallel to this plane, the perimeter of the cross-section is always the same.

Defined this way, it's obvious that cuboids, or more broadly parallelepipeds and cylinders, are untapered. But interestingly, regular tetrahedra and octahedra are also untapered. Moreover, there exists a rich class of untapered tetrahedra.

Since I started designing mechanical puzzles around 20 years ago, I have been consciously focusing on the intriguing aspect of untaperedness, even though I wasn't using that specific term at the time. In fact, a few years after I began, I successfully incorporated this concept into several puzzle designs. Among these, the one that stood out to me was based on the untaperedness of a degenerated tetrahedron.

However, I wasn't entirely satisfied with its realization in that form and continued to embrace the basic idea of untaperedness. About a decade after I started designing mechanical puzzles, I suddenly came up with the idea for this puzzle, though I've completely forgotten the direct trigger. To be precise, the idea of using a credit card-sized card to make it easily replicable by anyone occurred to me a bit later. Initially, what I had realized was a mathematical discovery: any rectangular plate could be used to create essentially the same puzzle, provided a bag of suitable dimensions is prepared.

I don't introduce the solution to Card in the Bag in this article, but the essence of its solution is included in Japanese patent P5131793. Those who are interested might want to consult it.

7. Conclusion

When I came up with Card in the Bag (or, more accurately, an essentially the same puzzle), I felt even more strongly than with my other puzzle creations why such a puzzle hasn't existed before and that it wouldn't have been surprising if it had been invented long ago. The level of satisfaction of this feeling was truly gratifying.

On the other hand, when I thought calmly, I felt it's understandble why this puzzle hadn't been thought of before. The puzzle, once created, seems to be just a combination of very familiar materials, and it looks unrelated to any specialized mathematical concepts like untaperedness. However, for me to conceive this puzzle, I had to go through the concept of untaperedness (even though I didn't use that specific term). Moreover, contemplating that idea for an extended period was likely essential. When I eventually found a way to realize it without any atmosphere of complicated concepts, the result was, as I describe it with modesty aside, a masterpiece that appears as if it could have existed before long, yet had remained undiscovered.

All the puzzles I've discussed in this article seem to share something in common with what I've just described. Each began with an idea that I found very interesting from my own mathematical aesthetic perspective. Then, whether it took a long time or not, when I was fortunate enough to realize that this idea could be implemented through something very simple and commonplace, then, a "masterpiece" of a puzzle was born.

I would like to mention two final points. First, as I stated at the outset, I hope my descriptions have conveyed some of the delights of devising new creations. It would be extremely gratifying if this could serve as a reference or inspiration in any way.

Second, I have a request. If you have any ideas that could be interesting if transformed into a puzzle, but you have no idea how to make it happen, please do share them with me. I would value such concepts, contemplating them and hoping to one day turn them into fascinating puzzles.

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