Jeu de Taquin: The Fifteen Puzzle in research mathematics

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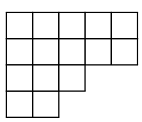
February 2024

In French the *Fifteen Puzzle* [Sl06] is known as *Jeu de Taquin* ("The teasing game"). It inspired the French mathematician Schützenberger [Sch61] to create combinatorial *jeu de taquin*, which has somewhat different rules. The sliding of numbers around each other turns out to be a very useful tool in algebraic combinatorics, with applications to representation theory and algebraic geometry.

Schützenberger [Sch72, Sch73] and others later generalized much of the theory from labelings of squares in the plane to any finite partially ordered set. See the survey [Sta09] by Stanley for more on the history and for further references. Here we just explain a few fun accessible facts about *jeu de taquin* (in the research sense, which is the main usage of this phrase in English, since we already have a name for the "fifteen puzzle"). The references provide a starting point for more on the history, details, and proofs.

Our slides take place in left- and top-justified

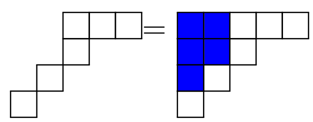
straight shapes



where the rows weakly decrease in length

or

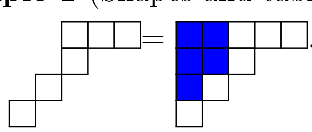
skew shapes



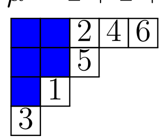
which are the difference of two straight shapes.

Putting numbers in the squares of our shape so that they are **always** increasing from top-to-bottom and from left-to-right turns it into a tableau. For any skew shape σ with n squares, we denote by $\mathrm{SYT}(\sigma)$ the set of all tableaux of shape σ with exactly the numbers $\{1, 2, \ldots, n\}$, each used once.

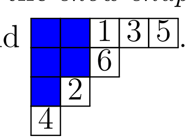
Example 1 (Shapes and tableaux)**.** For $\lambda = 5 + 3 + 2 + 1$ and $\mu = 2 + 2 + 1$, the *skew shape* λ/μ is



. Two *tableaux* of this shape are



and



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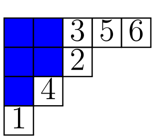
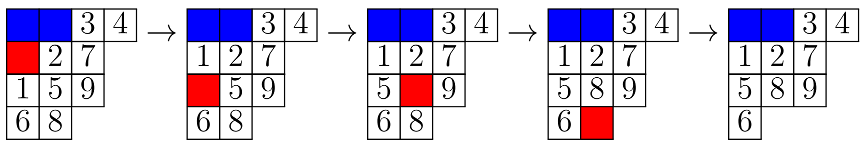
But



is **not** a tableaux because the 2 and 3 are out of order.

The next example shows some valid jeu de taquin slides. For mathematical jeu de taquin, once an empty square has been identified, each following individual slide is deterministic because of the ordering constraint. This is unlike the fifteen puzzle, where slides can result in elements being out of order, allowing more than one slide to be legal into an empty square.

Example 2 (Jeu de taquin move from one skew tableau to another)**.** Here we show the succession of individual slides that takes the chosen red square from the NW boundary of the skew shape to the SE boundary. At each step, the *smaller* of the adjacent squares (below or to the right) slides into the red square, and the red (empty) square moves to the original location of that smaller number. We call this a jdt move.

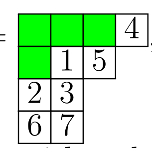


Note that every sliding move preserves the property that numbers increase from top to bottom and from left to right (though the intermediate tableaux are not necessarily of skew shape). The tableau on the right is the result of jeu de taquin applied to the initial skew shape and the selected red square. In some cases, we want to track where the red square ended, in which case the penultimate tableau better represents the output. Both points of view are useful.

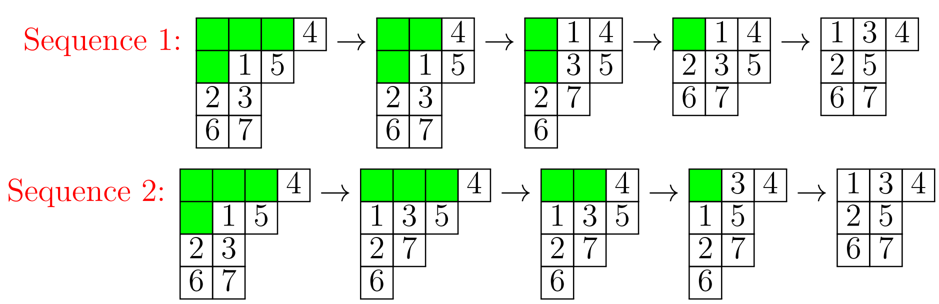
From the mathematical point of view amazing things happen!

Rectification of skew tableaux

One natural (and useful) procedure is to use jeu de taquin to move a skew tableaux to a straight shape, a process called rectification. Even though each jeu de taquin move is deterministic, there is in general more than one order in which the moves can be performed to reach a straight shape.

Example 3 (Rectifying a skew tableau)**.** If we start with the tableau: Q = , then

our first jdt move can begin with a slide into the rightmost green square in either the first row, or in the second row. Here are two (of the three possible) sequences of jeu de taquin slides.



Note the that at the penultimate step, we have different tableaux (of distinct shapes), even though we have done jdt moves into the same *set* of boxes. The next theorem says this cannot happen if we slide all the way to a straight shape (so rectification is well defined).

Theorem 1 (Confluence)**.** *Any sequence of jeu de taquin moves leading to a* ***straight*** *shape gives the same result (regardless of the order of the moves).*

The above examples illustrates this theorem.

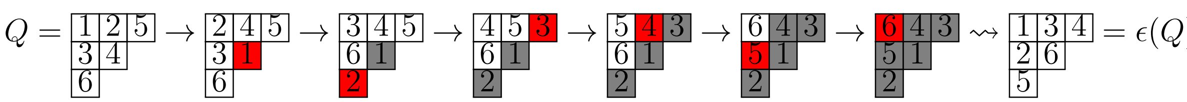
Lauren K. Williams of Mercyhurst University has a number of useful applets, including one that animates this type of jeu de taquin slide.

https://www.integral-domain.org/lwilliams/Applets/discretemath/jeudetaquin.php

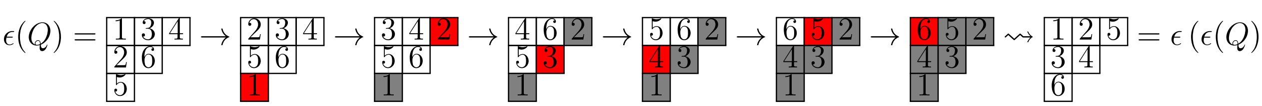
Evacuation of tableaux

Iterating jdt moves leads to an operation on skew tableaux called evacuation. We treat the lowest numbered square as empty, perform a jdt move, and keep track of where it ends up on the boundary. We then "freeze" that square and entry, so it does not participate in successive slides. Repeat this with the new lowest-numbered square, and continue until each square in the shape has undergone jeu de taquin (in the decreasing shapes). In the last step, reinterpret the result as a tableau (by reversing the ordering of the labels). This gives a map $\epsilon : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$.

Example 4 (Evacuating a tableau)**.** For the straight tableau Q below we compute $\epsilon(Q)$ using six successive jeu de taquin moves. The red squares indicate the square that has just undergone a jdt move, and the gray squares are frozen.



If we repeat this procedure on $\epsilon(Q)$, we get back to our original shape. The next theorem shows that this is not a coincidence.



Theorem 2 (Evacuation is an involution)**.** *For any tableau Q, $\epsilon(\epsilon(Q)) = Q$, so $\epsilon^2 = \mathrm{id}$.*

This theorem continues to hold for skew-shaped tableaux. In fact, the sliding process, jeu de taquin moves and evacuation can all be generalized to labelings ("linear extensions") of *any* partially ordered set, and evacuation continues to be an involution [Sta09, Thm 2.1].

Promotion

Another (related) operation on $Q \in \mathrm{SYT}(\lambda)$ via jeu de taquin is called promotion. Perform a jdt move starting at the box with the lowest label until it gets to the boundary, where it becomes the new largest element. Then decrement the other labels by 1 to get the tableau $\partial(Q) \in \mathrm{SYT}(\lambda)$.

Example 5 (Promotion operation on a tableau)**.** Entries in red below show the sliding path for the jdt move.

$$Q = \begin{ytableau} *(red) 1 & 3 & 6 \\ 2 & 4 & 9 \\ 5 & 7 \\ 8 \end{ytableau} \to \begin{ytableau} \textcolor{red}{2} & 3 & 6 \\ \textcolor{red}{4} & \textcolor{red}{7} & 9 \\ 5 & *(red) 1 \\ 8 \end{ytableau} \rightsquigarrow \begin{ytableau} 1 & 2 & 5 \\ 3 & 6 & 8 \\ 4 & 9 \\ 7 \end{ytableau} = \partial(Q) \to \begin{ytableau} \textcolor{red}{2} & \textcolor{red}{5} & \textcolor{red}{8} \\ 3 & 6 & *(red) 1 \\ 4 & 9 \\ 7 \end{ytableau} \rightsquigarrow \begin{ytableau} 1 & 4 & 7 \\ 2 & 5 & 9 \\ 3 & 8 \\ 6 \end{ytableau} = \partial^2(Q)$$

A natural question is how large is the period of this map on $\mathrm{SYT}(\lambda)$. That is, what is the minimum number of times we need to apply ∂ that guarantees we end up where we started, no matter what tableau we start with? In general the orbit structure is quite disregulated, and the period of promotion is quite large, as the next example shows.

Example 6 ([SW12])**.** Promotion has order 7,554,844,752 on $\mathrm{SYT}(\lambda)$ for $\lambda = \ydiagram{8,6}$, which has only 14 squares.

But for certain special shapes, the order of promotion is exceptionally small.

Theorem 3 (Order of promotion on rectangular tableaux)**.** *Promotion on* $\mathrm{SYT}(a \times b)$ *has order* ab*.*

Example 7 (Promotion orbits for tableaux of rectangular shape)**.** For the 5 tableaux of shape 2×3, we get one orbit of size three, and one of size two, so the order of promotion is $\mathrm{LCM}(2,3) = 6 = 2 \cdot 3$, agreeing with the theorem.

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 & 6 \end{ytableau} \to \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 & 6 \end{ytableau} \to \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 & 6 \end{ytableau} \Lsh \qquad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 & 6 \end{ytableau} \to \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 & 6 \end{ytableau} \Lsh$$

For the 14 tableaux of shape 4×2, we get orbits of size eight, four, and two, so the order of promotion is $\mathrm{LCM}(2,4,8) = 8 = 4 \cdot 2$.

$$\begin{ytableau} 1 & 3 \\ 2 & 6 \\ 4 & 7 \\ 5 & 8 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 & 6 \\ 7 & 8 \end{ytableau} \to \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 & 7 \\ 6 & 8 \end{ytableau} \to \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 & 6 \\ 7 & 8 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 & 7 \\ 6 & 8 \end{ytableau} \to \begin{ytableau} 1 & 4 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{ytableau} \to \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 & 6 \\ 7 & 8 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 & 7 \\ 6 & 8 \end{ytableau} \Lsh$$

$$\begin{ytableau} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{ytableau} \to \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 7 & 8 \end{ytableau} \to \begin{ytableau} 1 & 3 \\ 2 & 4 \\ 5 & 7 \\ 6 & 8 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{ytableau} \Lsh \qquad \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 & 7 \\ 6 & 8 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{ytableau} \Lsh$$

There are a few other special shapes where the order of promotion is small [Sta09, Thm 4.1].

1 Bibliography

References

[Fom] Sergey Fomin, *Knuth Equivalence, Jeu de Taquin, and the Littlewood–Richardson Rule* in Richard P. Stanley, *Enumerative Combinatorics, Volume 2, Ed. 2*, Cambridge Univ Press, 2024, pp. 687–713.

[Hai] M. D. Haiman, *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math. **99** (1992), 79–113. https://doi.org/10.1016/0012-365X(92)90368-P

[Sch61] M.-P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Canad. J. Math. **13** (1961), 117–128.

[Sch72] M.-P. Schützenberger, *Promotion des morphismes d'ensembles ordonnés*, Discrete Math. **2** (1972), 73–94. https://doi.org/10.1016/0012-365X(72)90062-3

[Sch73] M.-P. Schützenberger, *Evacuations*, in Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo I, Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976, pp. 257–264.

[Sl06] Jerry Slocum and Dic Sonnevald *The 15 Puzzle Book: How it Drove the World Crazy*, Slocum Puzzle Foundation (2006).

[Sta09] Richard P. Stanley, *Promotion and Evacuation*, Electron. J. Combin. **16(2)** (2009), #R9, https://doi.org/10.37236/75.

[SW12] Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. **33** (2012), 1919–1942, https://doi.org/10.1016/j.ejc.2012.05.003