Jekyll and Hyde Chess Tours

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Here are some puzzles for G4G15 based on an upcoming paper for the 2024 Bridges conference titled "Artsy Pseudo Hamiltonian Tours" [2]. We define a k-Hamiltonian tour or cycle of a graph to be a sequence of the vertices of the graph, each vertex appearing once, such that each successive vertex in the sequence is graphical distance k from the previous vertex, and such that the end vertex in the sequence is the same as the starting vertex. For the cycle C_n, if k and n are relatively prime, a k-Hamiltonian tour is also known as the $\{n/k\}$ star polygon consisting of one path that visits each vertex once. $\{n/k\}$ is known as the Schläfli notation. An (a,b)-step Hamiltonian tour or cycle of a graph is a sequence of the vertices starting and ending at the same vertex such that the distances between successive vertices alternate between a and b, see Figure 1(b), where we extend the Schläfli notation to $\{n/(a,b)\}$. See [4]. Such designs sometimes appear in a variety of art forms, for example, Figures 1(c) and (d). Figure 1(d) duplicates the design of a three-person string loop octahedron. Each vertex is held by one hand; see a video at [1,"String Quartet"] showing how it is created. This could also be a design for six dance partners who each move through each of the six positions, in which under curves precede over curves, or for juggling patterns.

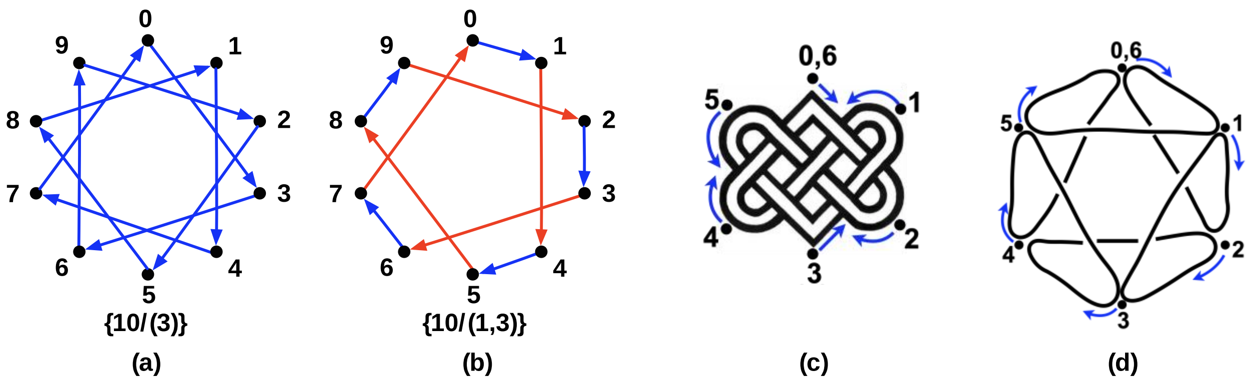


Figure 1: *(a) The {10/3} star polygon. (b) The {10/(1,3) tour, or the (1,3)-step Hamiltonian tour of* C_{10}*. (c) Celtic knot could represent {6/(3,2,2,3,-2,–2)} pattern; or six dancers following arrows, undercurves preceding over curves. (d) {6/(1,2)} string figure creating the octahedron or representing a dance or juggling pattern.*

Figure 2 : *Lace shirtwaist buttons by Laurel Shastri exhibiting several of the {m/(a,b)} patterns.*

However, for G4G15, it seems that a puzzle based on these ideas might be appealing. Can we, for example, find a tour of the 8 by 8 chess board by a chess piece that alternates its moves between those of two standard pieces? There are lots of possibilities, but here are some puzzles in which the touring piece alternates between a bishop and a knight. We'll call that piece the Jekyll and Hyde chess piece. Because the knight alternates colors on a checkerboard and the bishop stays on the same color, a tour can only work on a board with a multiple of four squares. See Figure 3, in which the first move is a knight move. If the first move is a bishop move, the number of squares must similarly be a multiple of four.

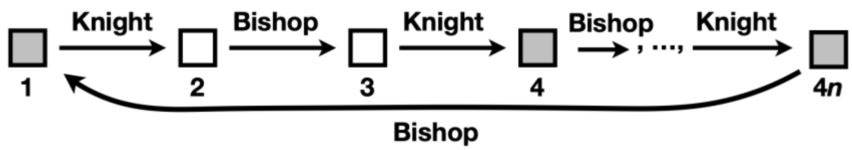
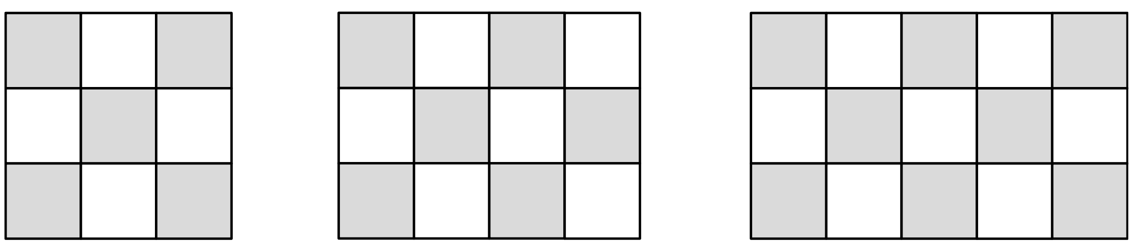


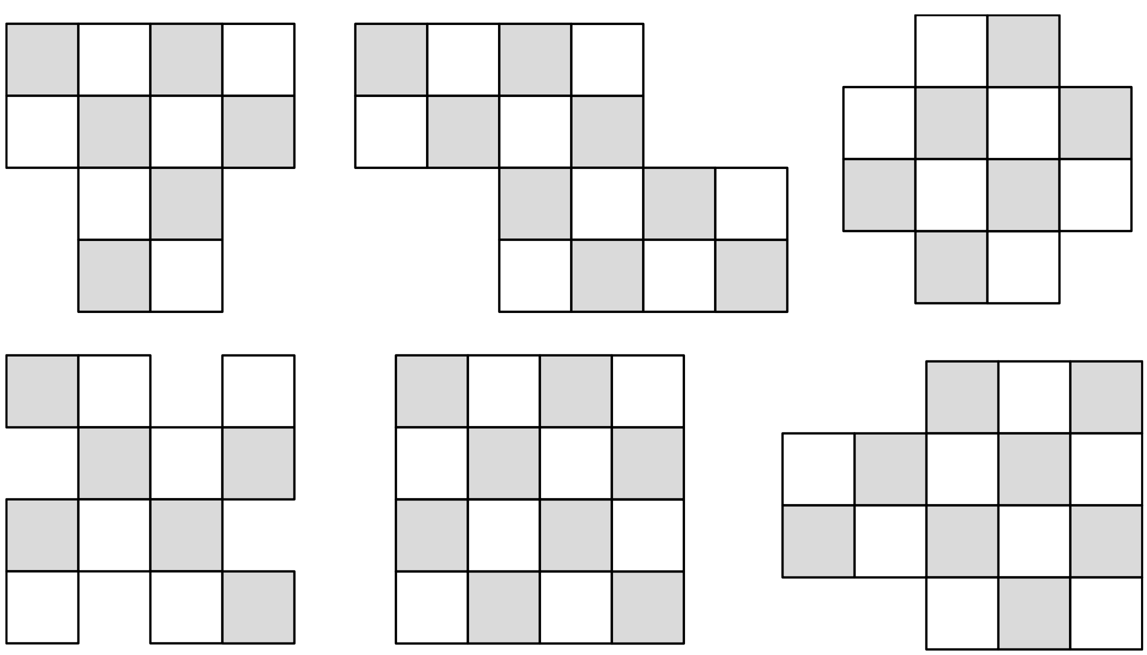
Figure 3: *A Jekyll and Hyde (alternating bishop and knight moves) tour of a checkerboard colored board must have a number of squares that is a multiple of four.*

If the board is composed of a number of squares that is not a multiple of four we might instead try to find a pseudo Hamiltonian path that visits every square but does not return to the starting square. The first few puzzles below are easier. Do try to find a Jekyll and Hyde knight/bisohop Hamiltonian tour of the 8 by 8 chessboard. Example solutions (I have not attempted to tabulate all solutions for each puzzle!) are at [3]. Make up some similar puzzles of your own. For example, try a Jekyll and Hyde piece that alternates knight and king moves. Or what about a piece that alternates the moves of three pieces (an (a,b,c)-step Hamiltonian tour).

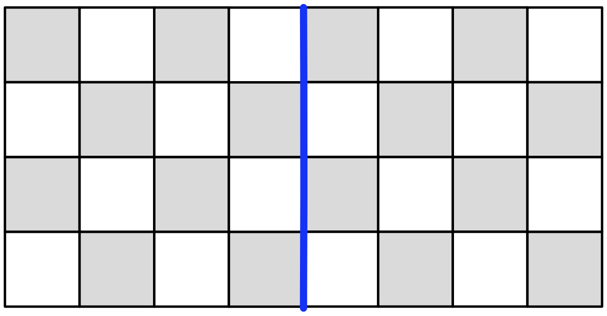
The 3 by 3 square below has a Jekyll and Hyde (bishop and knight) Hamiltonian path, but not a tour. The 3 by 4 rectangle has a genuine alternating Hamiltonian tour. The 3 by 5 rectangle has what we might call an "improper" tour, since it has an odd number of squares but there is a tour that starts with a knight move and ends in the starting square with a knight move rather than a bishop move. To record your solution you might write 1 in the starting square, 2 in the next etc. If your first move is a knight move then all the even numbers will be in a square that is the target of a knight move, and the bishop moves will end in odd numbered squares. Example solutions are at [].

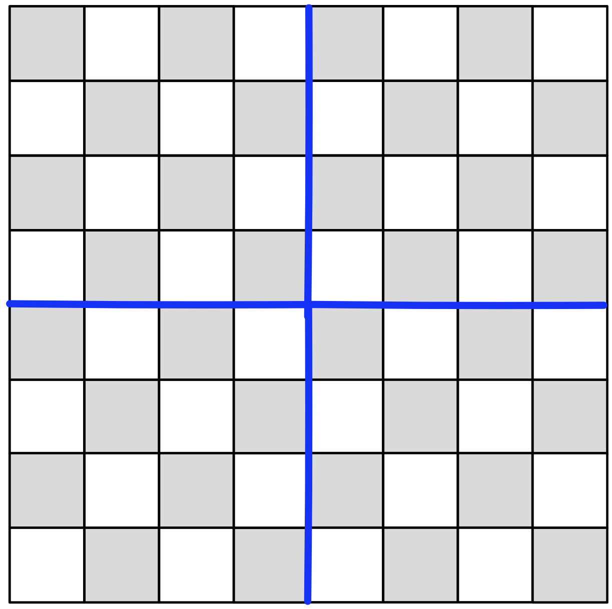


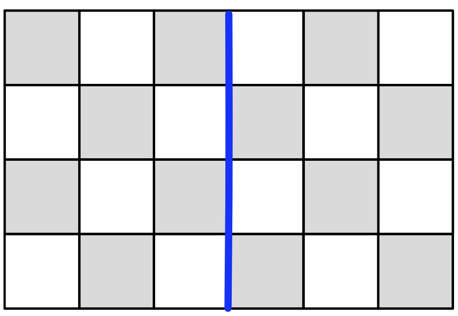
The following all have genuine Hamiltonian Jekyll and Hyde tours.

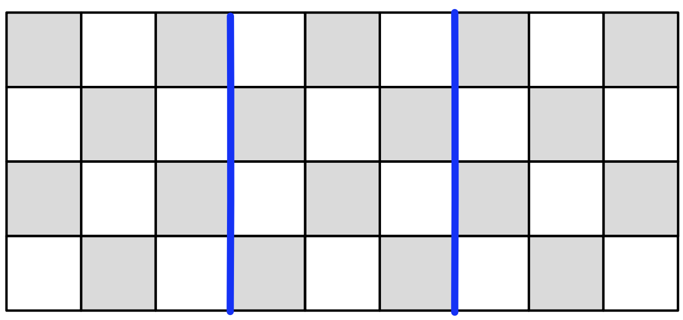


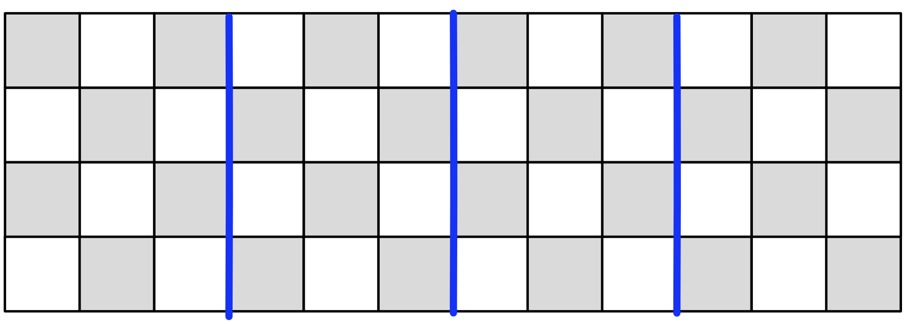
Can you use your solution to the 4 by 4 to find a solution to the 4 by 8 and the 8 by 8? You might be able to use your solution to the 3 by 4 rectangle to solve the 6 by 4 rectangle. Can you use the 3 by 4 and the 6 by 4 to extend and find a solution to the 9 by 4, in fact to all $3k$ by 4 rectangles?
The 3 by 7 has an improper tour.

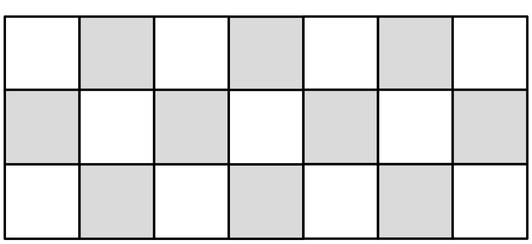












References.

[1] G. Csicsery. "Short films on string polyhedra with Scott Kim and Karl Schaffer." 2009. http://mathdance.org/html/CsicseryShortFilms.html.
[2] K. Schaffer and M. Nathan. "Artsy Pseudo Hamiltonian Tours." Bridges Richmond. 2024. Submitted.
[3] K. Schaffer. Solutions to these puzzles!. G4G15. 2024. https://www.movespeakspin.com/pdf/2024JekyllHydeChessSolutions.
[4] G.-C. Lau, S.-M. Lee, K. Schaffer, S-M Tong. "On (a,b)-step Hamiltonian Graphs. In progress. https://movespeakspin.com/pdf/2024_01_31_a_b__Step_Hamiltonian_Graphs.pdf.