Can the Number of Pieces in a Rectangular Jigsaw Puzzle be a Multiple of the Number of Edge pieces?

By D. Scott Hewitt 2/2/2022

Scott496@comcast.net

The short answer is yes. There are an infinite number of solutions. However, some interesting results appear in connection to this question. I occasionally do Jigsaw puzzles, and I usually work on the edge pieces first. That's how I became interested in this topic.

Note: If you are new to jigsaws, there are quite a few options for doing Jigsaw puzzles. There are many available in stores or online. More choices are available online when you do an internet search for software. Some are free and some require the purchase of a license. I like the option of creating my own puzzles from jpg files, which some Jigsaw companies offer.

Let's start with some definitions and formulas:

Let x = the number of pieces on the shorter side.

let y = the number of pieces on the longer side.

let e = the number of edge pieces.

let f = the ratio of total pieces to edge pieces.

Let d = y – x (the difference between the long side and the short side).

Now let t = x y = total pieces.

Number of edge pieces = e = 2x + 2y – 4 (true for any rectangular jigsaw puzzle)

Then f = $\frac{t}{e}$

We are interested in positive integer solutions for f.

We can assume that x > 1 because if x = 1 we just get a single row of pieces which is trivial.

Before talking about solutions, we have a few theorems in order to hopefully eliminate some possibilities: Most of the theorems are fairly basic and the proofs are short.

Theorem 1 is somewhat trivial and concerns jigsaw puzzles with only 2 rows of pieces.

<u>Theorem 1</u> If and only if x = 2 and y is any positive integer, then f = 1 and t = e

Proof: t = 2y and e = 2*2 +2y -4 = 2y

So f = $\frac{t}{e} = \frac{2y}{2y} = 1$, and every piece in the puzzle is an edge piece.

Note: If x > 2, then of course t > e.

Theorem 2 If x = 3, There is no solution for integer f.

Proof: t = 3y, and f = $\frac{t}{e} = \frac{3y}{6+2y-4} = \frac{3y}{2y+2}$

Since y ≥3, this fraction f takes on values on the interval $(\frac{9}{8}, \frac{3}{2})$ as y takes on values from 3 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 3.

Theorem 3 If x = 4, there is no solution for integer f.

Proof: t = 4y, and f = $\frac{t}{e} = \frac{4y}{2(4+y)-4} = \frac{4y}{2y+4} = \frac{2y}{y+2}$

Since y ≥ 4, this fraction f takes on values on the interval $[\frac{4}{3}, 2)$ without reaching 2, as y takes on values from 4 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 4.

Theorem 4 x and y cannot both be odd.

Proof: If x and y are both odd then t is odd. We have f = $\frac{t}{2x+2y-4}$

The numerator is odd and the denominator is even, so x and y both odd is not possible.

Theorem 5 x and y cannot be equal, so square jigsaw puzzles are not possible with integer f.

Proof: t = x^2

$$f = \frac{t}{e} = \frac{x\hat{}2}{4x-4}$$

Case I

In the case of an even square where x = 2m, we have $\frac{4m\hat{}2}{8m-4} = \frac{m\hat{}2}{2m-1}$

If this fraction is equivalent to an integer, then k(2m-1) = m^2

We have: m^2 -2km+k = 0 so m = k+$\sqrt{k(k-1)}$

For m to be an integer k(k-1) must be a perfect square.

However, consecutive integers do not share any factors. Therefore, k and k-1 must be consecutive perfect squares which is impossible.

Case II

In the case where x = 2m + 1, we have f = $\frac{4m^2+4m+1}{8m}$, but this is also impossible since an even number cannot divide an odd number. Therefore, x and y cannot be equal.

Theorem 6 y cannot be a multiple of x.

Proof: Suppose y = ax, then t = xy = ax^2

e = 2(x+ax) – 4 or e = x(2+2a) – 4

We now have f = $\frac{x*x*a}{x(2+2a)-4}$ and we know that x ≥ 5 from previous work.

If x = 5 we have f $= \frac{25a}{5(2+2a)-4}$ and a number of the form 25r or 5r cannot be divisible by a number of the form 5r - 4.

If x = 6 we have f = $\frac{36a}{6(2+2a)-4}$ and again, a number of the form 36r or 6r cannot be divisible by a number of the form 6r – 4

For x ≥ 7 let us look at the general case f = $\frac{x*x*a}{x(2+2a)-4}$ The numerator is of the form xr and the denominator is of the form xr – 4 . Numbers of the form xr – 4 can never divide numbers of the form xr with x ≥ 7.

Therefore, for all x ≥ 5, f cannot be an integer. This proves that y cannot be a multiple of x.

Now let us look at a condition that does yield solutions. Suppose x and y have a difference of 2.

Theorem 7 If d = y - x = 2 there are an infinite number of solutions.

Proof: If y - x = 2 then t = x(x+2) and e = 2x +2(x+2) – 4 = 4x

We have f = $\frac{x(x+2)}{4x} = \frac{x+2}{4}$ and we can solve for any f.

Example A: Suppose we wish the number of edge pieces to be exactly half the total number of pieces.

So, f is 2 and we have $\frac{x+2}{4} = 2$

X+2 = 8

X = 6 and therefore y =8

t = 48 and e = 24

Example B: suppose we wish f to be 3.

We have $\frac{x+2}{4} = 3$

X + 2 = 12

X = 10 and therefore y = 12 giving us t = 120 and e = 40

Example C: suppose we wish f to be 73.

We have $\frac{x+2}{4} = 73$

X + 2 = 292

X = 290 and y =292 giving us t = 84680 and e = 1160

The following table gives a solution for f =2 through 20 where y - x = 2. Incidentally, these are also the jigsaws of smallest area with those f values. Other solutions exist for these f values when y – x > 2.

Table 1

f = t/e	x = width	y = length	t = total pieces	e = edge pieces
2	6	8	48	24
3	10	12	120	40
4	14	16	224	56
5	18	20	360	72
6	22	24	528	88
7	26	28	728	104
8	30	32	960	120
9	34	36	1224	136
10	38	40	1520	152
11	42	44	1848	168
12	46	48	2208	184
13	50	52	2600	200
14	54	56	3024	216
15	58	60	3480	232
16	62	64	3968	248
17	66	68	4488	264
18	70	72	5040	280
19	74	76	5624	296
20	78	80	6240	312

Table 2 gives the solutions for jigsaw puzzles of minimum area with consecutive x values where x = width. Notice that when x is a prime such as 31 or 43, or when x is an odd square like 49; y is sometimes quite large compared to neighboring y values due to the more difficult task of finding a solution with integer f.

Table 2

X = width	Y= length	t = x y	e = edge	f = t/e
5	12	60	30	2
6	8	48	24	2
7	30	210	70	3
8	18	144	48	3
9	14	126	42	3
10	12	120	40	3
11	24	264	66	4
12	20	240	60	4
13	132	1716	286	6
14	16	224	56	4
15	26	390	78	5
16	42	672	112	6
17	36	612	102	6
18	20	360	72	5
19	306	5814	646	9
20	27	540	90	6
21	38	798	114	7
22	24	528	88	6
23	48	1104	138	8
24	44	1056	132	8
25	92	2300	230	10
26	28	728	104	7
27	50	1350	150	9
28	65	1820	182	10
29	60	1740	174	10
30	32	960	120	8
31	870	26970	1798	15
32	50	1600	160	10
33	62	2046	186	11
34	36	1224	136	9
35	44	1540	154	10
36	68	2448	204	12
37	150	5550	370	15

38	40	1520	152	10
39	74	2886	222	13
40	57	2280	190	12
41	84	3444	246	14
42	44	1848	168	11
43	1722	74046	3526	21
44	90	3960	264	15
45	86	3870	258	15
46	48	2208	184	12
47	96	4512	282	16
48	92	4416	276	16
49	282	13818	658	21
50	52	2600	200	13

From table 1 and table 2, we see that every x value greater than or equal to 5 yields at least one solution, and every f value greater than or equal to 2 yields multiple solutions.

Tables 3 and 4 are just to show that perfect squares and cubes are possible for the number of edge pieces.

Table 3 lists very specific cases where e is a perfect square and $x \leq 100$. Although there appears to be an infinite number of solutions, there are only 5 solutions for x below or equal to 100. If we were to go a bit further, there are 18 solutions for x below 500.

Table 3

x	y	t = x y	e	f = t/e
18	56	1008	144	7
50	152	7600	400	19
98	296	29008	784	37
98	1472	144256	3136	46
100	2352	235200	4900	48

Table 4 lists cases where e is a perfect cube and x is less than 1000.

Table 4

x	y	t = x y	e	f = t/e
54	56	3024	216	14
162	704	114048	1728	66
250	252	63000	1000	63
686	688	471968	2744	172

Triangular numbers are numbers of the form T(n) = $\frac{n(n+1)}{2}$. If we ask whether both x and y can be triangular numbers, the answer is yes; but solutions seem to be somewhat scarce. Table 5 gives the only 10 solutions I could find with the restriction that both x and y are less than 100,000. I was unable to find a formula to calculate these directly. It's interesting that both T88 and T168 show up twice.

Table 5

x	y	t=x*y	e=edge	f	Triangular Index for x	Triangular Index for y
78	171	13338	494	27	T12	T18
903	1378	1244334	4558	273	T42	T52
1770	24310	43028700	52156	825	T59	T220
2850	3916	11160600	13528	825	T75	T88
3916	5253	20570748	18334	1122	T88	T102
11325	14196	160769700	51038	3150	T150	T168
14196	17578	249537288	63544	3927	T168	T187
26106	31375	819075750	114958	7125	T228	T250
52003	60726	3157934178	225454	14007	T322	T348
81406	93528	7613740368	349864	21762	T403	T432

Now let us examine differences between y and x. Using a computer program, I did a search and found solutions for every difference up to d = 250 with the exception of d = 1, 3, 4, and 6.

Table 6 gives minimum solutions for all known consecutive values of d = y – x up to 25.

Table 6

d = y-x	x = short side	y = long side	t = total pieces	e = edge pieces	f = $\frac{t}{e}$
2	6	8	48	24	2
5	9	14	126	42	3
7	5	12	60	30	2
8	12	20	240	60	4
9	35	44	1540	154	10
10	8	18	144	48	3
11	15	26	390	78	5
12	30	42	1260	140	9
13	11	24	264	66	4
14	18	32	576	96	6
15	104	119	12376	442	28
16	14	30	420	84	5

17	21	38	798	114	7
18	32	50	1600	160	10
19	17	36	612	102	6
20	24	44	1056	132	8
21	209	230	48070	874	55
22	10	32	320	80	4
23	7	30	210	70	3
24	132	156	20592	572	36
25	23	48	1104	138	8

I call d values which do not yield a solution "difference outlaws".

Hypothesis(A): d = 1, 3, 4, and 6 are difference outlaws.

Hypothesis(B): d = 1, 3, 4, and 6 are the <u>only</u> outlaws. Therefore, all other differences are possible. Some differences yield x values that are quite large. For example, d = 225 yields the solution x = 25199, y = 25424, and f = 6328 with no smaller solution.

If someone would like to work on hypothesis A or B, I can get you started:

d = 1 → f = $\frac{x^2+x}{4x-2}$ → x^2 - 4xf +x + 2f = 0

d= 3 → f = $\frac{x^2+3x}{4x+2}$ → x^2 - 4xf +3x – 2f = 0

d = 4 → f = $\frac{x^2+4x}{4x+4}$ → x^2 - 4xf +4x – 4f = 0

d = 6 → f = $\frac{x^2+6x}{4x+8}$ → x^2 - 4xf +6x – 8f = 0

The fractions for f are equivalent to the Diophantine equations on the right that we would like to prove have no solution for x and f as positive integers. My feeling is that these 4 Diophantine equations probably have integer solutions but that the solutions involve negative integers. Incidentally, these equations represent hyperbolic curves.

Let's take a last look at one of the outlaws: d = 1. This looks like it may be easier than d = 3, 4, or 6.

Theorem 8

X and y cannot be consecutive integers. In other words, $d \neq 1$.

Proof: Suppose d = 1, then we have f = $\frac{x(x+1)}{4x-2}$

Suppose we treat this as a quadratic equation and solve it for x. We have

x^2 +(1-4f)x + 2f = 0

Using the Quadratic Formula: x = $\frac{(4f-1)\pm\sqrt{(1-4f)^2-8f}}{2}$

To have a solution, the discriminant must be a perfect square: $(1\text{-}4f)^2 - 8f$ = n^2

We have 1 – 16f + 16f^2 = n^2 and n must be odd for x to be a positive integer.

The left side of the equation can be written as $16(f^2\text{-}f) +1$ and this is equivalent to $8(2f^2 \text{-}2f) +1$

This looks encouraging since all odd squares must be of the form 8n+1. In fact, all odd squares are of the form 8T + 1 where T is a triangular number of the form $\frac{r(r+1)}{2}$

The question now is: Can $2f^2$ -2f ever be a triangular number?

If so, we have $2f^2$ -2f = $\frac{r(r+1)}{2}$ (standard form for triangular numbers)

$4f^2$ -4f – r(r+1) = 0 and we must again resort to the quadratic formula.

f = $\frac{(4\ )\pm\sqrt{4^2+16r(r-1)}}{8} = \frac{4\pm 4\sqrt{r(r+1)+1}}{8}$

Looking at this fraction, r(r+1) must be equivalent to 8T = 8 $\frac{v(v+1)}{2}$ = 4v(v+1), and we are missing a factor of 4. We have encountered a contradiction, so jigsaw puzzles with integer f are not possible if y – x = 1.

Feel free to send me an email if you have a comment about my paper or if you have made progress on Hypothesis A or B that you would like to share.

In conclusion, I would like to wish you an enjoyable time working on your next jigsaw puzzle, regardless of whether or not the number of pieces is a multiple of the number of edge pieces! Have fun!