ROOT(Math Success) = Childhood + Concrete Analogs + Challenges

Debbie Denise Reese, PhD
Zone Proxima LLC
ceo@zoneproxima.com

The current USA landscape for mathematics achievement is bleak. Nationwide assessment in 2022 reported 73% grade 8 and 64% of grade 4 students lacked mathematics proficiency (as in structure mapping U.S. Department of Education et al., 2023). Proficiency deficiency persists across grade levels. Recent data are only the latest in a national trend over decades (see also, National Center for Educational Statistics (NCES), 2022): the majority of our young people from early years through high school graduating class lack mathematics proficiency. Our students lack the skills to succeed in "typical modern economy jobs" (Siegler et al., 2012, p. 691). Effective concrete, early childhood mathematics preparation is a formative step toward mathematics and lifelong achievement (Mullis et al., 2020b; National Association for the Education of Young Children (NAEYC) & National Council of Teachers of Mathematics (NCTM), 2002; National Council of Teachers of Mathematics, 2022). Large scale research studies have found that early childhood math readiness

- Predicts high school math achievement (Watts et al., 2014)
- Predicts adult socioeconomic status (i.e., adult age 42, Ritchie & Bates, 2013).

The Zone Proxima Math initiative for early childhood (Reese, 2022) was designed to address the need for effective early childhood mathematics.

Origins of Zone Proxima Math

Zone Proxima Math derived from a Metaphorics analysis of the issues in USA mathematics education. Metaphorics is applied analogical reasoning theory. Analogical reasoning is a fundamental, ubiquitous cognitive process (Holyoak & Thagard, 1995; Hummel & Holyoak, 1997). People learn by matching relational structure from a concrete, relatively familiar domain to an unfamiliar one (Gentner, 1983), leading to inferences about the abstract or unfamiliar domain. The process is automatic, with low cognitive overhead. Immediate goals and context guide selection of analogs and mapping (Gentner & Holyoak, 1997). Analogical reasoning is recognized as a key to scientific and mathematical learning and discovery (e.g., Gentner, 1980; Kuhn, 1993; Polya, 1954). Metaphorics applies analogical reasoning structure mapping and pragmatics constraints to the design (especially domain specification), development, assessment, and evaluation of instruction (Reese, 2003, 2009, 2015a). Fundamentally, novice learners should manipulate effectively designed and sequenced concrete experiences. These concrete analogs should share profound relational structures (a deep system composed of layered and branched relational connections) with the to-be-learned domain. The learning environment should produce challenges as goals that motivate and orient the learner to discover viable connections within the domain of the concrete analogs. This will produce learner construction of viable mental models of foundational concepts in a domain such as mathematics (see also, Reese, 2015c). Related new learning, such as transfer from the concrete analog to more abstract symbols, becomes intuitive. When learning is intuitive, learners have a greater chance of success.

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Low-risk (Reese, 2010, 2015a, 2015b), well-designed, embodied learning challenges can effectively guide young children to discover and apply foundational mathematics concepts. During guided discovery learning, children can experience and learn to recognize joy in mathematics thinking and activity. Given that early childhood learning in the home and within other child care environments produces associated gains in mathematics achievement (Mullis et al., 2020a, 2020b), Zone Proxima teaches **adults how to effectively mentor, recognize, and reward guided mathematics discovery and application by guiding their young children**:

- to manipulate concrete learning objects designed to embody mathematical concepts,
- to recognize and guide children to self-regulate (includes tenacity),
- to engage in metacognition while engaged in mathematical problem-solving, thinking and activity,
- to enjoy the rewards of effortful cognitive engagement (hard work can be good fun).

Zone Proxima children construct the integrated, coherent, well-formed mental models of foundational math required for mathematical understanding and success. The Zone Proxima Math vision and intervention for academic and personal flourishing derives from (a) over 25 years of learning science research and development, (b) fieldwork identifying the USA mathematics proficiency need, and (c) fieldwork applying cognitive science to mathematics instruction. Zone Proxima Math instruction works the way the mind does: leveraging cognitive recognition between the abstract and the concrete.

What's Metaphoric About Zone Proxima Math?

Zone Proxima Math employs pedagogical tools like 5-frames, ten-frames, counters, subitizing, tangrams, and pattern blocks as expertly described in the editions of the Van de Walle mathematics pedagogy textbook (e.g., Van de Walle et al., 2016). Zone Proxima Math employs other sources as well: for additional number sense, e.g., Jo Bohler's youcubed program (Boaler & youcubed Team, 2024), learning trajectories (e.g., Clements & Sarama, 2021), growth mindset (e.g., Dweck, 1989; Dweck et al., 2011), metacognition (e.g., Flavell, 1979; National Research Council Committee on the Foundations of Assessment et al., 2001, see p. 78; Schoenfeld, 2004), and flow (Csikszentmihalyi & Csikszentmihalyi, 1988; Reese, 2010, 2015b; as well as the related construct of flourishing, see Zhang, 2022). But the core Metaphorics component of Zone Proxima Math is the concrete analog of Cuisenaire Rods (invented by Emile-Georges Cuisenaire) coupled with the early childhood mathematics curriculum for using the Rods with guided discovery as developed by Caleb Gattegno (1970), and Zone Proxima evaluation, approach, and enhancements see Figure 1.

Although theorists and educators have extolled the virtues of Cuisenaire Rods, many lack the solid grounding of why and how the Rods are such a powerful learning tool. This might explain why many Cuisenaire Rods inhabit educators' classroom closets with but rare expeditions into the light for discrete "fun" or "hands-on" activities. Rather, the Rods should be core components of on-going, coherent infusion within the school year's curriculum over many grades (at least preschool through middle school). Before John Holt's crusade promoting homeschooling and unschooling, he extolled the virtues of Cuisenaire Rods:

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> The beauty of the Cuisenaire Rods is not only that they enable the child to discover, by himself, how to carry out certain operations, but also that they enable him to satisfy himself that these operations really work, really describe what happens. (1964, p. 78)

Figure 1. Children ages 4 to 5.5 during Zone Proxima Math sessions (mother, child, and a virtual—via Zoom—Zone Proxima mentor: mother and mentor not pictured) using Cuisenaire Rods, a modified Gattegno Math curriculum, and Zone Proxima learning aids. Copyright 2024 by Zone Proxima LLC.

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Yet, Holt's published writings present strong evidence that he did not understand—or at least apply—the true power of the Cuisenaire Rod analog: the deep relational mapping from the Rods' physical characteristics to mathematical relationships. For example, consider one section of the same book in which Holt described his fifth-grade students and lessons using the Rods as **counters** rather than relationally rich analogs. He directed the student to divide quantities of Rods into cups. Placing the Rods into "containers" should always privilege the learner's concrete interaction with length x width x depth. Cups obscure any physical properties of the individual and/or combined rods. The mathematical relational properties are infused into the physical characteristics of the Rods. Holt's implementation removed the power of the analog. The implementation removed the relational correspondence between the analog and multiples, factors, and equivalence.

Like many educators —be they parents, caregivers, or teachers—Holt seems unaware of the shortcomings of his implementation of the Cuisenaire Rod learning technology.[i] Holt wrote

> This work has changed most of my ideas about the way to use Cuisenaire rods [*sic*], and other materials. It seemed to me at first that we could use them as devices for packing in recipes[ii] much faster than before, and many teachers seem to be using them this way. But this is a great mistake. What we ought to do is use these materials to enable children to make for themselves, out of their own experience and discoveries, a solid and growing understanding of the ways in which numbers and the operations of arithmetic work. Our aim must be to build soundly, and if this means that we must build more slowly, so be it. Some things we will be able to do much earlier than we used to—fractions, for example. Others, like long division, may have to be put off until later. The work of the children themselves will tell us. (p. 120)

I agree with Holt that guided discovery learning is powerful instructional approach. So, too, are apt concrete instructional learning objects.

An apt instructional analog is a relationally dense and aligned in one-to-one correspondence with a targeted learning domain. Primary concepts and terminology are

- Mapping (as defined within structure mapping theory, Gentner, 1983): the cognitive process of placing two domains into correspondence according to their shared relational structure. Based upon the source (concrete or familiar) domain, the analogizer (learner) infers the existence of to-be-learned relations in the target (to-be-learned) domain. Metaphorics designers of instruction work in reverse, specifying the targeted to-be-learned domain and then designing a relationally isomorphic concrete analog (source domain), goal structures, and learning context designing the source domain (e.g., here the Cuisenaire Rods and early childhood lessons). This "backward analogizing" is the essence of Metaphorics (Reese, 2015a, 2015c).
- Systematicity: the degree of relational density within a domain and within the mapping from the source (concrete or familiar) domain to the to-be-learned domain. Apt instructional analogs embody deep systematicity with the targeted learning domain (Gentner, 1983; Reese, 2009).

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- Isomorphic: a one-to-one relational correspondence between the concrete (or familiar) domain and the to-be-learned domain (Gentner, 1983; Reese, 2009).
- Embodied: (Reese, 2015c): When humans use their senses and perceptual processing to interact with the physical (real or virtual), those experiences are embodied. Seymore Papert (1980) coined the term body syntonicity for this condition: human learning when it is related to "individuals' sense and knowledge about their bodies" (p. 63). The more that interaction corresponds to the individual's sense of the individual's body and its cause and effect with the physical world, the greater its body syntonicity, the greater the syntonicity, the more natural the learning experience and analogical mapping from the source domain to the targeted learning domain.[iii]
- Pragmatic constraints: An analogizer's situation-specific goal structures dictate (constrain (i.e., constrain, see Spellman & Holyoak, 1992; Spellman & Holyoak, 1996) selection of analog domains and the mappings between the domains (Holyoak et al., 2001). During game-based learning, a game's goal structure can provide the necessary constraint to guide analog selection and cross-domain mapping (e.g., Reese, 2012). During Zone Proxima Math, "challenges" —which are the session-level (lesson-level) problems posed to learners within guided discovery learning—serve as the pragmatic constraint.

Memorization and rote application of algorithms or formulas leads to discrete, disconnected, isolated, and inert knowledge. Today's math education quandary has produced a significant proportion of youth with inert, disconnect mathematics knowledge. When these individuals do attempt mathematics, they often do not know what algorithm to use, when/why to use it, if the memory of the algorithm is correct, and if calculated results are viable. Meaningful knowledge is connected knowledge. A human can hold 7 ∓ 2 chunks of knowledge in working memory (cognitive load theory further posits a limit of only 2 or 3 active nodes within working memory, see Paas et al., 2003; Reese et al., 2016; Sweller et al., 1998). However, each chunk may contain an infinitely large and expandable knowledge network. Expert knowledge contains chunks of large, integrated knowledge networks. This is why experts can bring so much accrued knowledge to bear on a problem. Knowledge networks lessen cognitive load. The goal of sound education—and the Zone Proxima Math enterprise—is to guide children to discover, apply, and build dense knowledge networks of viable (see von Glasersfeld, 1995 for discussion of human knowledge and viability) domain knowledge. Instruction should guide learners to construct "robust, cohesive, and normative views" (Linn et al., 2004, p. 37).

Powerful instructional and learning technologies must be implemented appropriately. It is incumbent that the educator implementing an apt concrete learning object (i.e., an apt, manipulative concrete analog) appropriately apply the pedagogical content knowledge (Shulman, 1986) that supports its instructional application. Without the pedagogical content knowledge that informs instructional delivery, inadequately executed the lessons are, at best, a waste of instructional/learning time. At their worst implementation, such lessons may produce intractable misconceptions—doing more harm than good.

Metaphorics to the Rescue

Zone Proxima Math serves a mission: to rectify early childhood math instruction and ready youth for mathematics success that will follow them through academic preparation and through professional and personal lives. Zone Proxima provides instruction to the adults who care for and about young children (see Figure 2). Zone Proxima mentors train adults, during sessions for adults and sessions for adult-child dyads, to effectively and knowledgably guide young children to discover and apply foundational mathematics concepts. Typically, training involves one adult session per week and an adult-child dyad per week. Adults and their dyads may train individually or as part of a group class. Zone Proxima also provides professional development. Ideally, Zone Proxima Math and Cuisenaire Rods should integrate from early childhood through middle school education, with high school courses using the manipulatives to reactivate prior Cuisenaire Rod-related knowledge as necessary. Zone Proxima Math trains adults to provide effective mathematics education by leveraging what we know about the human mind. Zone Proxima Math is designed to work. . . the way the mind does.

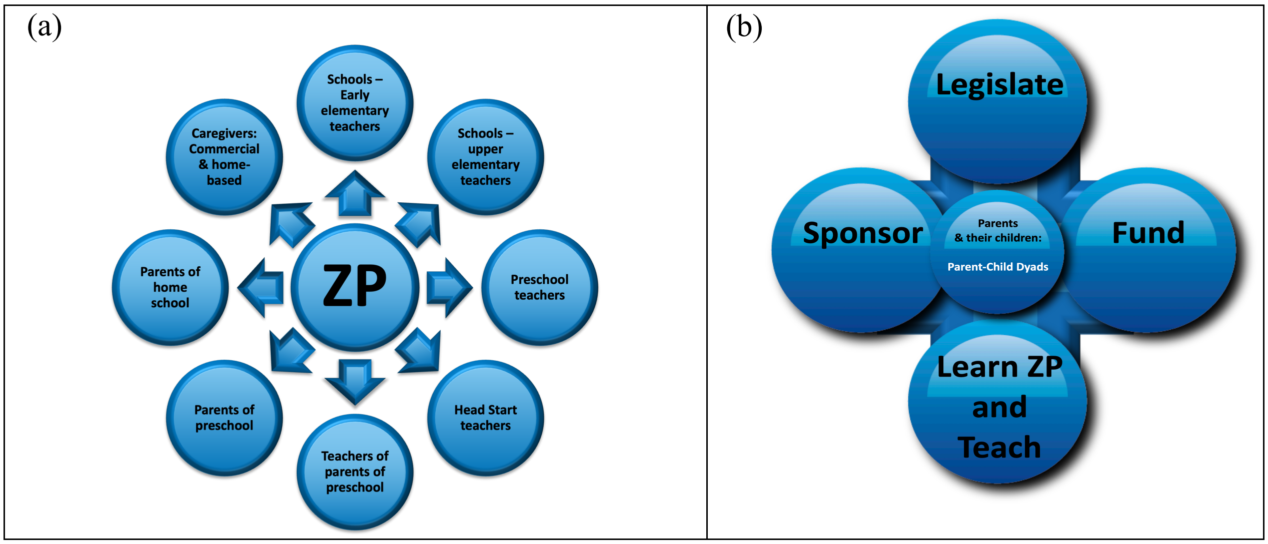

Figure 2. (a) Zone Proxima flexibly provides and designs training for educators working with early childhood in diverse environments. (b) This call to action (a) recruits adults who train to provide effective early childhood mathematics education, (b) those who might sponsor/fund trainings in populations lacking financial resources to pay for training, and (c) those who might propose, advise, and legislate Zone Proxima Math solutions for early childhood mathematics achievement.

References

Boaler, J., & youcubed Team. (2024). *youcubed*. Stanford Graduate School of Education. https://www.youcubed.org

Clements, D. H., & Sarama, J. (2021). *Learning trajectories*. LearningTrajectories.org. Retrieved January 14 from https://www.learningtrajectories.org/documents/1626808776340.pdf

Csikszentmihalyi, M., & Csikszentmihalyi, I. S. (Eds.). (1988). *Optimal experience: Psychological studies of flow in consciousness*. Cambridge University Press.

Dweck, C. S. (1989). Motivation. In A. Lesgold & R. Glaser (Eds.), *Foundations for a psychology of education* (pp. 87-136). Erlbaum.

ROOT(Math Success) =

Dweck, C. S., Walton, G. M., & Cohen, G. L. (2011). *Academic tenacity: Mindsets and skills that promote long-term learning*. William and Melinda Gates Foundation. Retrieved from http://web.stanford.edu/~gwalton/home/Publications_files/DweckWaltonCohen_2014.pdf.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive–developmental inquiry. *American Psychologist*, *34*(10), 906-911. https://doi.org/https://doi.org/10.1037/0003-066X.34.10.906

Gattegno, C. (1970). *Gattegno mathematics: Text-book 1*. Educational Solutions

Gentner, D. (1980). *The structure of analogical models in science (report No. 4451, NTIS No. AD-A087-625)*. National Technical Information Service, U.S. Department of Commerce.

Gentner, D. (1983). Structure mapping: A theoretical framework for analogy. *Cognitive Science*, *7*, 155-170. http://groups.psych.northwestern.edu/gentner/papers/Gentner83.pdf

Gentner, D., & Holyoak, K. J. (1997). Reasoning and learning by analogy: Introduction. *American Psychologist*, *52*(1), 32-34.

Hestenes, D., Wells, M., & Swackhamer, G. (1992). Force concept inventory. *The Physics Teacher*, *30*, 141-158. https://doi.org/10.1119/1.2343497

Holt, J. (1964). *How children fail*. Dell Publishing Co, Inc. https://archive.org/details/howchildrenfail0000john/page/n5/mode/2up

Holyoak, K. J., Gentner, D., & Kokinov, B. N. (2001). Introduction: The place of analogy in cognition. In D. Gentner, K. J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 1-20). MIT Press.

Holyoak, K. J., & Thagard, P. (1995). *Mental leaps: Analogy in creative thought*. MIT Press.

Hummel, J. E., & Holyoak, K. J. (1997). Distributed representations of structure: a theory of analogical access and mapping. *Psychological Review*, *104*(3), 427-466.

Kuhn, T. (1993). Metaphor in science. In A. Ortony (Ed.), *Metaphor and thought* (2nd ed., pp. 533-541). Cambridge University Press.

Linn, M. C., Eylon, B.-S., & Davis, E. A. (2004). Knowledge integration perspective on learning. In M. C. Linn, E. A. Davis, & P. Bell (Eds.), *Internet environments for science education* (pp. 29-46). Lawrence Erlbaum Associates.

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020a). *Highlights: TIMSS 2019 international results in mathematics and science*. Boston College, TIMSS & PIRLS International Study Center. Retrieved February 14, 2024 from https://timss2019.org/reports/wp-content/themes/timssandpirls/download-center/TIMSS-2019-Highlights.pdf

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020b). *TIMSS 2019 international results in mathematics and science*. Boston College, TIMSS & PIRLS International Study Center. Retrieved February 14, 2024 from https://timssandpirls.bc.edu/timss2019/international-results/

National Association for the Education of Young Children (NAEYC), & National Council of Teachers of Mathematics (NCTM). (2002). *Early Childhood Mathematics: Promoting Good Beginnings*. National Association for the Education of Young Children. https://www.naeyc.org/sites/default/files/globally-shared/downloads/PDFs/resources/position-statements/psmath.pdf

National Center for Educational Statistics (NCES). (2022). *Trends in international mathematics and science study (TIMSS)* (2019; Institute for Education Science. http://nces.ed.gov/timss/idetimss/

ROOT(Math Success) =

National Council of Teachers of Mathematics. (2022). *Mathematics in early childhood learning*. https://www.nctm.org/Standards-and-Positions/Position-Statements/Mathematics-in-Early-Childhood-Learning-2146936947/

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

National Research Council Committee on the Foundations of Assessment, Board on Testing and Assessment, C. f. E., & Division of Behavioral and Social Sciences and Education. (2001). *Knowing what students know: The science and design of educational assessment*. National Academy Press. Retrieved from http://www.nap.edu/books/0309072727/html/. http://www.nap.edu/books/0309072727/html/

Paas, F. G. W. C., Renkl, A., & Sweller, J. (2003). Cognitive load theory and instructional design: Recent developments. *Educational Psychologist*, *38*(1), 1-4. http://search.ebscohost.com/login.aspx?direct=true&AuthType=ip&db=afh&AN=9110450

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Basic Books.

Polya, G. (1954). *Mathematics and plausible reasoning: Volume 1: Induction and analogy in mathematics* (Vol. 1). Princeton University Press.

Reese, D. D. (2003). *Metaphor and content: An embodied paradigm for learning* [Dissertation, Virginia Polytechnic Institute and State University]. Blacksburg, VA. http://scholar.lib.vt.edu/theses/available/etd-03312003-115151/unrestricted/Reese_D_D.pdf

Reese, D. D. (2009). Structure mapping theory as a formalism for instructional game design and assessment. In B. Kokinov, K. Holyoak, & D. Gentner (Eds.), *New frontiers in analogy research: Proceedings of the 2nd International Conference on Analogy (Analogy '09)* (pp. 394-403). New Bulgarian University Press. http://www.nbu.bg/cogs/analogy09/proceedings/42-T2.pdf

Reese, D. D. (2010). Introducing Flowometer: A CyGaMEs assessment suite tool. In R. V. Eck (Ed.), *Gaming & cognition: Theories and perspectives from the learning sciences* (pp. 227-254). IGI Global. https://doi.org/10.4018/978-1-61520-717-6.ch011

Reese, D. D. (2012). CyGaMEs: A full-service instructional design model harnessing game-based technologies for learning and assessment. In L. Moller & J. B. Huett (Eds.), *The next generation of distance education: Unconstrained learning* (pp. 157-170). Springer. https://doi.org/10.1007/978-1-4614-1785-9_10

Reese, D. D. (2015a). CyGaMEs *Selene* player log dataset: Gameplay assessment, flow dimensions, and non-gameplay assessments. *British Journal of Educational Technology*, *46*(5), 1005-1014. https://doi.org/10.1111/bjet.12322

Reese, D. D. (2015b). Dashboard effects challenge flow-learning assumption in digital instructional games. In S. Tettegah & W. D. Huang (Eds.), *Emotions, Technology, and Games* (pp. 231-287). Elsevier. https://doi.org/10.1016/B978-0-12-801738-8.00011-7

Reese, D. D. (2015c). Embodied learning systems. In J. M. Spector, T. Johnson, D. Ifenthaler, W. Savenye, & M. Wang (Eds.), *SAGE encyclopedia of educational technology* (pp. 259-262). SAGE. https://doi.org/10.4135/9781483346397

Reese, D. D. (2022). *Zone Proxima math*. Zone Proxima LLC. http://math.zoneproxima.com

Reese, D. D., Pawluk, D. T. V., & Taylor, C. R. (2016). Engaging learners through rational design of multisensory effects. In S. Tettegah & S. U. Noble (Eds.), *Emotions, Technology, and Design* (pp. 103-127). Elsevier. https://doi.org/10.1016/B978-0-12-801872-9.00006-5

ROOT(Math Success) =

Ritchie, S. J., & Bates, T. C. (2013). Enduring Links From Childhood Mathematics and Reading Achievement to Adult Socioeconomic Status. *Psychological Science*, *24*(7), 1301-1308. https://doi.org/10.1177/0956797612466268

Schoenfeld, A. H. (2004). The math wars. *Educational Policy*, *18*(1), 253-286. https://doi.org/10.1177/0895904803260042

Shulman, L. E. (1986). Those who understand: Knowledge growth in teachers. *Educational Researcher*, *15*(2), 4-14.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science*, *23*(7), 691-697. https://doi.org/10.1177/0956797612440101

Spellman, B. A., & Holyoak, K. J. (1992). If Saddam is Hiltler then who is George Bush? Analogical mapping between systems of social roles. *Journal of Personality and Social Psychology*, *62*(6), 913-933.

Spellman, B. A., & Holyoak, K. J. (1996). Pragmatics in analogical mapping. *Cognitive Psychology*, *31*(3), 307-346.

Sweller, J., van Merrienboer, J. J. G., & Paas, F. G. W. C. (1998). Cognitive Architecture and Instructional Design [Article]. *Educational Psychology Review*, *10*(3), 251-296. http://search.ebscohost.com/login.aspx?direct=true&AuthType=ip&db=afh&AN=965008

U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, & National Assessment of Educational Progress (NAEP). (2023). *NAEP Data Explorer*. National Center for Education Statistics. httpd://www.nationsreportcard.gov/ndecore/xplore/NDE

Van de Walle, J., Karp, K. S., & Bay-Williams, J. (2016). *Elementary and middle school mathematics: Teaching developmentally, enhanced Pearson eTest with loose-leaf version* (9th ed.). Pearson.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Routledge Falmer Press.

Watts, T. W., Duncan, G. J., Siegler, R. S., & Davis-Kean, P. E. (2014). What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educational Researcher*, *43*(7), 352-360. https://doi.org/10.3102/0013189X14553660

Zhang, Q. (2022). *Sensation, Experience, and Being Alive: Foundations of Cognition, Learning, and Flourishing* [PhD, Arizona State University]. Tucson, AZ.

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[i] Indeed, John Caldwell Holt's position as a leader in educational reform and the home schooling movement began with this publication.

[ii] Here, I assume Holt uses "recipes" to refer to the standard algorithms, as delineated in current Common Core standards (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010).

[iii] It is well established: a primary obstacle for fundamental, introductory knowledge acquisition in scientific disciplines is the misalignment with the physical phenomena such as the scale at which it occurs, human perception of their environment, and human interpretations of their experiences of cause and effect. (Hestenes, Wells, & Swackhamer, 1992).

Commonsense belief about how the physical world works derived from years of personal experiences. Many can be incompatible with complex concepts - such as those in introductory Newtonian physics or genetics. Specifically, it has been established that (1) commonsense beliefs about motion and force are incompatible with Newtonian concepts in most respects, (2) conventional physics instruction produces little change in these beliefs, and (3) this result is independent of the instructor and the mode of instruction. The implications could not be more serious. Since the students have evidently not learned the most basic Newtonian concepts, they must have failed to comprehend most of the material in the course. They have been forced to cope with the subject by rote memorization of isolated fragments and by carrying out meaningless tasks. No wonder so many are repelled! The few who are successful have become so by their own devices, the course and the teacher having supplied only the opportunity and perhaps inspiration. (p. 141)