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WHAT HO! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

FEBRUARY 2014

PROMOTIONAL CORNER: Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.

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PUZZLER: If you split a deck of shuffled cards into two piles of 26 cards each, then one pile is sure to contain a red card and the other a black card.

If you split the shuffled deck into four piles of 13 cards, it is always possible to select a spade from one pile, a diamond from a second pile, a heart from a third, and a club from the fourth. Why?

If one splits a shuffled deck into thirteen piles of 4 cards each and lay the piles face up, prove it is always possible to select an Ace, a 2, a 3, all the way up to a King, each from a different pile. (Try it! This makes for a fun game of solitaire.)

SEMI-MAGIC SQUARES

A square array of numbers is said to be *semi-magic* with magic number m if the entries in each row and each column of the array sum to m. For example, the following is a 3×3 semi-magic square of integers with magic sum 5.

$$\begin{pmatrix} 2 & 0 & 3 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{pmatrix}$$

(A square array is said to be *fully magic* if the entries of its two main diagonals sum to the same magic value as well.)

Question: What must be the magic sum of a semi-magic non-square rectangle?

If we think of semi-magic squares as matrices (see this month's CURRICLUM ESSAY), then we can perform arithmetic on these objects. The results are surprising!

Suppose A and B are semi-magic $k\times k$ matrices with magic sums m and n respectively. Then:

1. Their matrix sum $A+B$ is semi-magic with magic sum $m+n$.

2. If λ is a real number, then λA is semi-magic with magic sum λm.

3. Their product AB is also semi-magic with magic sum mn.

4. If A has an inverse, then A^{-1} is also semi-magic with magic sum $\frac{1}{m}$.

For example, for

$$A=\begin{pmatrix} 2 & 0 & 3 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{pmatrix}$$

with magic sum $m=5$, and

$$B=\begin{pmatrix} 0 & 2 & -1 \\ 1 & 3 & -3 \\ 0 & -4 & 5 \end{pmatrix}$$

with magic sum $n=1$, we have:

$$A+B=\begin{pmatrix} 2 & 2 & 2 \\ 3 & 7 & -4 \\ 1 & -3 & 8 \end{pmatrix}$$

which has magic sum $5+1=6$,

$$3A=\begin{pmatrix} 6 & 0 & 9 \\ 6 & 12 & -3 \\ 3 & 3 & 9 \end{pmatrix}$$

which has magic sum $3\times5=15$,

$$AB=\begin{pmatrix} 0 & -8 & 13 \\ 4 & 20 & -19 \\ 1 & -7 & 11 \end{pmatrix}$$

which has magic sum $5\times1=5$, and

$$A^{-1}=\frac{1}{20}\begin{pmatrix} 13 & 3 & -12 \\ -7 & 3 & 8 \\ -2 & -2 & 8 \end{pmatrix}$$

which has magic sum $\frac{4}{20}=\frac{1}{5}$.

Whoa!

Before reading on, try to explain why the property of being semi-magic is preserved by each of these matrix operations. (I find preservation under matrix product and matrix inverse particularly surprising!)

EXPLAINING THE MATRIX PROPERTIES

Let J be the $k \times k$ square matrix all of whose entries are 1. For example, for the 3×3 case,

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Take the semi-magic matrix A with magic sum 5 from the previous page. Notice that

$$AJ = \begin{pmatrix} 2 & 0 & 3 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} = 5J$$

and $JA = 5J$ too. In fact we see:

Theorem: *A square matrix A is semi-magic with magic sum m precisely if $AJ = mJ$ and $JA = mJ$.*

The properties 1, 2, 3, and 4 now follow from matrix arithmetic.

From $AJ = mJ = AJ$ and $BJ = nJ = JB$ we see:

Proof of 1:

$$(A+B)J = AJ + BJ = mJ + nJ = (m+n)J$$

$$J(A+B) = JA + JB = (m+n)J$$

Thus $A+B$ is semi-magic with magic sum $m+n$.

Proof of 2:

$$(\lambda A)J = \lambda(JA) = \lambda mJ$$

$$J(\lambda A) = \lambda(JA) = \lambda mJ$$

Thus λA is semi-magic with magic sum λm.

Proof of 3:

$$(AB)J = A(BJ) = nBJ = nmJ$$

$$J(AB) = (JA)B = mJB = mnJ$$

Thus AB is semi-magic with magic sum mn.

Proof of 4:

From $AJ = mJ$ we see $A^{-1}AJ = mA^{-1}J$, that is, $J = mA^{-1}J$, giving $A^{-1}J = \frac{1}{m}J$.

From $JA = mJ$ we obtain $JA^{=1} = \frac{1}{m}J$.

Thus A^{-1} is semi-magic with magic sum $\frac{1}{m}$.

> **Question:** If A is a semi-magic matrix that possesses a square root (that is, there is a matrix B with the property that $B^2 = A$) must the square root matrix also be semi-magic? (If so, with what magic sum?)

INTEGER SEMI-MAGIC SQUARES

From now on let's only consider square semi-magic matrices with non-negative integer entries.

The zero matrix (with all entries zero) is the simplest of these matrices. It has magic sum zero.

A semi-magic square of magic sum 1 has the property that each row and each column possesses exactly one entry non-zero entry, which is necessarily a 1. For example:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is semi-magic with magic sum 1.

> **Question:** There are six 3×3 semi-magic squares with magic sum 1. (Care to write them all down?) Show that, in

general, there are $n!$ $n \times n$ semi-magic squares with magic sum 1. Why do you think mathematicians choose to call these matrices *permutation matrices*?

We can form a semi-magic square of magic sum 2 by adding together any two permutation matrices. For example:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

Challenge: And conversely … Prove that any semi-magic matrix with magic sum 2 is a sum of two permutation matrices.

In general, we can create a semi-magic matrix of magic sum any non-negative integer value m we desire simply by adding together m permutation matrices.

Here is a remarkable result:

RESULT:
Any semi-magic matrix with non-negative integer entries and magic sum m is a sum of m permutation matrices.

We'll prove this result later in this essay. Actually, we'll prove the following pre-result first and use it to prove the result.

PRE-RESULT:
For any $k \times k$ semi-magic matrix with non-negative integer entries and magic sum m it is possible to circle k non-zero entries in the array with precisely one circled entry per row and one circled entry per column.

For example:

$$\begin{pmatrix} \textcircled{1} & 1 & 0 \\ 0 & 0 & \textcircled{2} \\ 1 & \textcircled{1} & 0 \end{pmatrix}.$$

EXPLAINING THE OPENING PUZZLER

Split a shuffled deck into four piles of 13 cards each. Look at the four piles, count how many cards of each suit lie in the piles, and fill out the following table please.

	Pile 1	Pile 2	Pile 3	Pile 4
# Spades				
# Diamonds				
# Hearts				
# Clubs				

As there are 13 cards of each suit in a deck the numbers in each row are sure to sum to 13. As each pile holds 13 cards the numbers in each column are sure to sum to 13. Thus the table you create is sure to be a semi-magic square with magic sum 13.

By the pre-result we can circle four non-zero entries in this table, one per row and one per column. These circled entries show which suit to pull from which pile to complete the puzzle.

Challenge: Actually take out those four cards, so that you now have four piles of 12 cards each. Explain why you can repeat the feat a second time – removing one card of each suit from four different piles. (Do it!) And then repeat the feat a third time, and a fourth time, and fifth time, and …!

If, instead, we split the shuffled deck into thirteen piles of four cards each …

This time draw a 13×13 table that displays the number of cards of each face-value in each pile. This gives a semi-magic square of magic sum 4, which must be a sum of four permutation matrices. This means it is possible to extract one card of each different face value from the 13 different piles AND do this four times in a row until there are no cards left! (Now that's a real game of solitaire. It really is fun to try to do this!)

PROOFS

Let's first prove the pre-result.

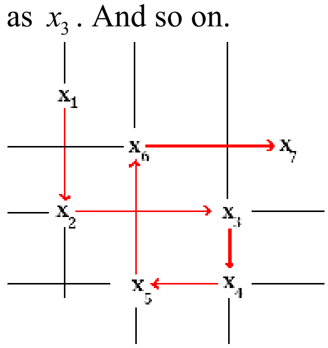
Suppose we have a $k \times k$ semi-magic square with non-negative integer entries and magic sum m.

If every non-zero entry in the matrix is m, then circle those entries: they must lie one per row and one per column.

Suppose, on the other hand, there is some non-zero entry $x_1 < m$ in the matrix. Then since the column in which x_1 lies has entries that sum to m there must be another non-zero entry $x_2 < m$ in the same column.

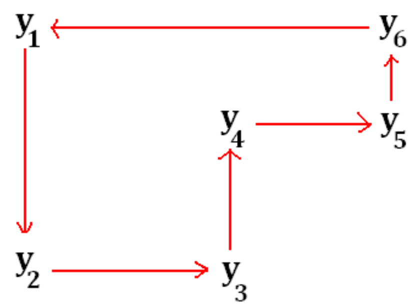
Since the row in which x_2 lies has entries that sum to m there must be a non-zero entry $x_3 < m$ in the same row as x_2.

Since the column in which x_3 lies has entries that sum to m, here must be a non-zero entry $x_4 < m$ in the same column as x_3. And so on.



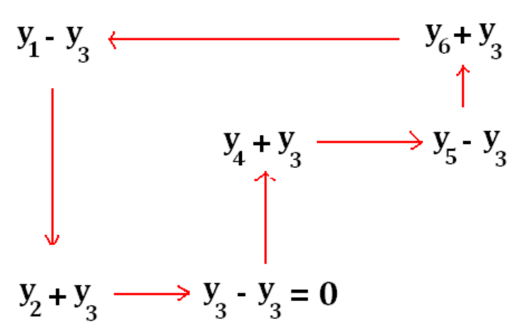
Eventually, after an even number of "steps," this process must return to column previously visited. This proves:

In a semi-magic square with magic sum m and not all non-zero entries of value m it is possible to find a "loop" of non-zero entries, call them $y_1 y_2 \ldots y_r$, all of value $< m$, so that y_1 and y_2 lie in the same column, y_2 and y_3 in the same row, y_3 and y_4 in the same column, and so on, all the way to y_r and y_1 in the same row.



Question: What does this mean for shuffled cards in four piles of thirteen and their suits?

One of these values has smallest value. Suppose, for example, in the diagram above y_3 is the smallest. Alternately subtract and add this smallest value to the entries in the loop. (No entry will be negative.)



This process doesn't change the row and column sums in the matrix but it has produced at least one additional matrix entry of zero.

We have:

In a semi-magic square with magic sum m and not all non-zero entries of value m, it is possible adjust some of the non-zero entries of the matrix so that it remains semi-magic and some additional zeros appear in the matrix.

We can repeat this process whenever there are entries of value $< m$ in the matrix, always creating new zeros in the matrix and only possibly adding value to entries that were originally non-zero.

Repeat this process until we can do so no further. This leaves a matrix will all non-zero entries equal to m, one per row and one per column. The locations of these entries of value m were non-zero entries in the original matrix. So we have found k non-zero entries in the original matrix, one per row and one per column, just as desired.

Proof of Main Result:

We want to prove that every semi-magic square of magic sum m is a sum of m permutation matrices.

This is true for $m = 1$: every semi-magic matrix of magic sum 1 is a permutation matrix.

In general, if A is a $k \times k$ semi-magic square of magic sum m, by the pre-result we can circle k non-zero entries in A, one per row and one per column. The locations of those circles correspond to a permutation matrix. Call that permutation matrix P.

Subtract 1 from each of the circled entries. This gives the matrix $A - P$, and it is still semi-magic, with magic sum $m-1$. If we are following an argument by induction, $A - P$ is thus a sum of $m-1$ permutation matrices:

$$A - P = Q_1 + Q_2 + \cdots + Q_{m-1}$$

It follows that A is thus a sum of m permutation matrices.

$$A = P + Q_1 + Q_2 + \cdots + Q_{m-1}.$$

Challenge: Consider a set of k students each to be assigned one of k distinct art projects. Each student writes a list of the projects she or he would like to do. My question: *When is it possible to assign projects so that each student is given a project on her or his list?*

For starters, each student must have at least one project written on her or his list. Among any two students there must be at least two different projects between them on their lists. And in general, among any r students there must be at least that many different projects listed among them.

a) Prove that if this condition is held then, among any set of r art projects there are at least that many different students who listed at least one of those projects.

b) Prove that if the condition is met, then it is possible to assign each student a project that is on her or his list. (So this condition is not only necessary, but also sufficient.)

[Look up *Hall's Matching Theorem* on the internet for so much more on this!]

Research Corner: Study $k \times k \times k$ semi-magic cubical arrays!

Are semi-magic cubes with non-negative integer entries sums of "permutation cubes" of some kind?

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