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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

September 2016

THIS MONTH'S PUZZLER:

Consider the numbers $\{0,1,2,3,4\}$ and their rearrangement $\{4,3,2,1,0\}$. Adding these two lists together termwise gives $\{4,4,4,4,4\}$, nothing but the square number four.

$$\{0,1,2,3,4\} + \{4,3,2,1,0\} = \{4,4,4,4,4\}$$

Adding $\{0,1,2,3,4,5\}$ and its rearrangement $\{0,3,2,1,5,4\}$ termwise, gives $\{0,4,4,4,9,9\}$, nothing but square numbers.

$$\{0,1,2,3,4,5\} + \{0,3,2,1,5,4\} = \{0,4,4,4,9,9\}$$

For each counting number N is there sure to be some rearrangement $\{0,1,2,...,N\}$ so that adding the two lists together termwise gives nothing but square numbers?

SQUARE PERMUTATIONS

In 1978, in his paper "Square permutations" (*Mathematics Magazine, Vol 51, No 1, January 1978, pp 64-66*) Benjamin Schwartz introduced the idea of the opening puzzle. (He said the idea is originally due to David Silverman.) Do square permutations always exist?

If we omit zero from our counting numbers, the answer is certainly no. For example,

No permutation of $\{1\}$ adds to give squares.

No permutation of $\{1,2\}$ adds to give squares.

No permutation of $\{1,2,3,4\}$ adds to give squares.

No permutation of $\{1,2,3,4,5,6\}$ adds to give squares.

No permutation of $\{1,2,3,4,5,6,7\}$ adds to give squares.

No permutation of $\{1,2,3,...,11\}$ adds to give squares.

(To see why no permutation exists for the final example, for instance, consider which elements we need to add to 4 and to 11 to make perfect squares.)

One can check that appropriate permutations of $\{1,2,3,...,N\}$ exists for all other values up to 11.

$\{1,2,3\}+\{3,2,1\}$
$\{1,2,3,4,5\}+\{3,2,1,5,4\}$
$\{1,2,3,4,5,6,7,8\}+\{8,7,6,5,4,3,2,1\}$
$\{1,2,3,4,5,6,7,8.9\}+\{8,2,6,5,4,3,2,9,1,7\}$
$\{1,2,3,4,5,6,7,8,9,10\}+\{3,2,1,5,4,10,9,8,7,6\}$

How about for values greater than 11? Can we construct a permutation that works for $N=12$?

Here's one possible strategy. Think of 16, the first square number larger than 12. Adding 4 to 12 gives 16 and we can use this to generate a partial list of termwise sums that work.

	4	5	6	7	8	9	10	11	12
+	12	11	10	9	8	7	6	5	4
	16	16	16	16	16	16	16	16	16

We're missing the terms 1, 2, and 3, but we know a permutation that works just for those three! Combining, we get a solution for $N=12$.

	1	2	3	4	5	6	7	8	9	10	11	12
+	3	2	1	12	11	10	9	8	7	6	5	4
	4	4	4	16	16	16	16	16	16	16	16	16

This technique fails, alas, for $N=13$. Adding 3 to 13 gives us the next square number, suggesting we work with a permutation of $\{3,4,,...,13\}$.

	3	4	5	6	7	8	9	10	11	12	13
+	13	12	11	10	9	8	7	6	5	4	3
	16	16	16	16	16	16	16	16	16	16	16

But we cannot juxtapose this with a working permutation of $\{1,2\}$.

Nonetheless, some trial-and-error shows that $N=13$ does have a valid square permutation.

$\{1,2,3,...13\}+\{8,2,13,12,11,10,9,1,7,6,5,4,3\}$

One can check by hand, by using the technique described above (which sometimes works) or by trial-and-error, that there are valid permutations for all N between 12 and 20.

Can we go higher?

Well, let's understand when the technique described works and when it fails. Here again is the picture that worked for $N=12$.

	1	2	3	4	5	6	7	8	9	10	11	12
+	3	2	1	12	11	10	9	8	7	6	5	4
	4	4	4	16	16	16	16	16	16	16	16	16

We are looking for a permutation of $\{1, 2, 3, ..., N\}$ that gives only square values upon termwise addition. Let k^2 be the smallest square number not smaller than N. (So $N \le k^2$.)

	1	2	3	k^2-N-1	k^2-N	...	N
+	*	*	*	*	N	...	k^2-N
		...			k^2	...	k^2

If we know there is a permutation that works for $\{1, 2, ..., k^2 - N - 1\}$, then we'll have a permutation that works for $\{1, 2, ..., N\}$.

What are the restrictions on k?

We need $k^2 - N$ to be a number no more than N, so we must have $k^2 - N \le N$, that is, $k^2 \le 2N$.

We also need $k^2 - N - 1$ to be among the values we know do have a permutation. We know that these exist for a whole string of values from 12 onwards. So let's see if we can arrange for $k^2 - N - 1$ to be ≥ 12, that is, for $k^2 \ge N + 13$.

Alright, for each value of $N \ge 12$, is it true that there is a square number k^2 with $N + 13 \le k^2 \le 2N$?

No! At least not for $N = 12$, 13, 14, 15, 16, or 17. Bother!

But might it hold for $N = 18$ and above? (This would be fine, as we know there are permutations that work from $N = 12$ to $N = 20$.)

So here is our refined question:

If $N \ge 18$ and k^2 is the smallest square number satisfying $N + 13 \le k^2$, does it then follow that $k^2 \le 2N$?

Well, let's see.

Suppose k is the smallest integer satisfying $k^2 \ge N + 13$, that is, satisfying $k \ge \sqrt{N+13}$. Then $k = \left\lceil \sqrt{N+13} \right\rceil$, and so k is no larger than $\sqrt{N+13} + 1$. Thus

$$\begin{aligned} k^2 &\le \left(\sqrt{N+13} + 1\right)^2 \\ &= N + 13 + 2\sqrt{N+13} + 1 \\ &= N + 14 + 2\sqrt{N+13}. \end{aligned}$$

We are hoping this is smaller than $2N$. So now we are wondering: *Is $14 + 2\sqrt{N+13}$ less than N?* That is, we are wondering if

$$2\sqrt{N+13} \le N - 14,$$

that is, if

$$4(N+13) \le N^2 - 28N + 196,$$

that is, if

$$0 \le N^2 - 32N + 144,$$

that is, if

$$0 \le (N-16)^2 - 112.$$

This is the case if we have $N - 16 \ge \sqrt{122}$, which we do if $N \ge 28$.

Ooh! That's a different range than $N \ge 18$!

TAKING STOCK

I am starting to get lost in details, in particular, the fine details that don't quite fit together. What do we have so far?

* We know that there are valid square permutations of $\{1, 2, ..., N\}$ for $N = 12$ up to $N = 20$. (They can be checked by hand.)

* We know that if we can find a square number k^2 that fits this table,

	1	2	3	k^2-N-1	k^2-N	...	N
+	*	*	*	*	N	...	k^2-N
		...			k^2	...	k^2

then we have a square permutation of $\{1, 2, ..., N\}$.

* By "fits this table" we mean that $k^2 - N \le N$ and that a valid square permutation for $\{1, 2, ..., k^2 - N - 1\}$ is already known to exist.

* We know that if $N \ge 28$, then there is square number k^2 with $k^2 - N \le N$ and with $k^2 - N - 1$ at least 12.

This feels like enough to prove

Theorem: *For each* $N \ge 12$ *there is a valid square permutation of* $\{1, 2, ..., N\}$.

Proof: On can check by hand that square permutations exist for all values $N = 12$ up to $N = 28$.

Here's how to construct a square permutation for $N = 29$.

Choose the smallest square number $k^2 \ge N + 13$. Then combine the known square permutation for $\{1, 2, ..., k^2 - N - 1\}$ (and here $k^2 - N - 1$ is a value somewhere between 12 and $N - 1$) with the permutation $\{N, ..., k^2 - N\}$ of $\{k^2 - N, ..., N\}$. One then has a square permutation of $\{1, 2, ..., N\}$.

	1	2	3		$k^2 - N - 1$	$k^2 - N$	$\cdots$	N
+	*	*	*		*	N	$\cdots$	$k^2 - N$
				$\cdots$		k^2	$\cdots$	k^2

Repeat this construction to get a square permutation of $N = 30$, then of $N = 31$, and so on, indefinitely.

CHALLENGE: Solve the opening puzzler. Prove that square permutations of $\{0, 1, 2, ..., N\}$ exist for all values of N. (This time there are no exceptional cases.)

CHALLENGE: Prove that for each $N \ge 1$ there is a permutation of $\{1, 2, ..., N\}$ so that each termwise sum is a prime number.

Ben Schwartz proves this first challenge in his paper and discusses ideas for proving the second.

RESEARCH CORNER

Explore other types of permutations of $\{1, 2, ..., N\}$ or of $\{0, 1, 2, ..., N\}$. Are there ones so that each and every termwise sum is a cube or just some odd power? Each is a triangular number? Each is a congruent to 0 or 1 modulo 8? And so on.

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