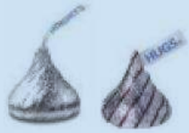
Julia Robinson Mathematics Festival®

A Sample of Mathematical Puzzles

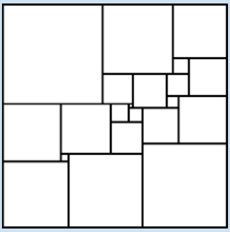
Julia Robinson
(1919 - 1985)



Hugs & Kisses

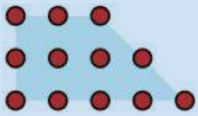
thesmartkitchenblog.com

Squareable Numbers

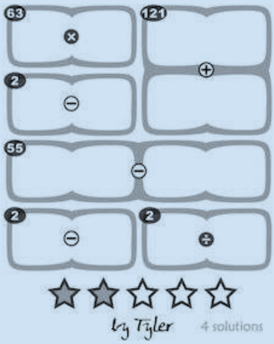


www.MathPickle.com

Squaring Puzzles



Trapezoidal Numbers



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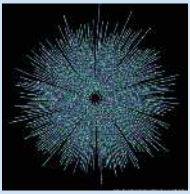
Cartouche Puzzles



www.itsokaytobesmart.com

Digit Sums

Switching Light Bulbs

"The JRMF really gets it right. Usually the best parts of mathematics are kept away from the public, as if you needed to be a mathematician to get to the fun stuff! It's refreshing to see a festival that brings this stuff to light, and in such a relaxed atmosphere. If you're lucky enough to have a JRMF near you, don't miss it! It's the best math party around."

– Vi Hart, Mathemusician, youtube.com/user/ViHart

Festival activities are designed to open doors to higher mathematics for students in grades K–12. Visit **www.JRMF.org** for more information about Julia Robinson Mathematics Festivals.

Compiled by Nancy Blachman, Founder, Julia Robinson Mathematics Festival.

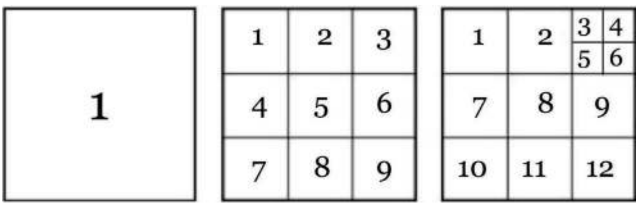
Your mom has three containers of candy for your summer treats! One container has hugs, one has kisses, and one has both. But, your little brother changed all the labels—he's even told you that every single label is WRONG. He'll let you pick one bag, and pick out one candy (blindfolded). Then, you have to figure out which candy is in which container. This chart may help you figure it out.

		Label SAYS:		
		"Kisses"	"Hugs and Kisses"	"Hugs"
In the containers				
				
				

Squareable Numbers

by Daniel Finkel and Katherine Cook, Math for Love

The number n is “squareable” if it is possible to build a square out of *n* smaller squares (of any size) with no leftover space. The squares need not be the same size. For example, 1, 9, and 12 are all squareable, since those numbers of squares can fit together to form another square.



Is there a simple way to tell if a number is squareable or not?

Which numbers from 1 to 30 are squareable? Experiment. Every time you come up with a way to break a square into some number of squares, circle that number.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

Is there a pattern? Can you predict squareability in general?

Here’s why Dr. Finkel proposed this problem to Gary Antonick, who published it in the New York Time Numberplay online blog, wordplay.blogs.nytimes.com/2013/04/08/squareable.

> *I think this puzzle is amazing because it’s compelling right away, and you can work on it without worrying too much about wrong answers. If you’re trying to show 19 is squareable and can’t, maybe you’ll accidentally show 10 is squarable on the way. (Of course, neither of those numbers is necessarily squareable. No spoilers here.) It’s great to be able to experiment with a puzzle in an environment where virtually everything you do gives you some positive gains.*
>
> *I also like it because the willy-nilly approach most people start with eventually leads to a more strategic approach, and it takes a combination of deeper strategies to solve the problem. I also like it because just about anyone can get started on it, and make some serious headway —you don’t need a sophisticated math background.*

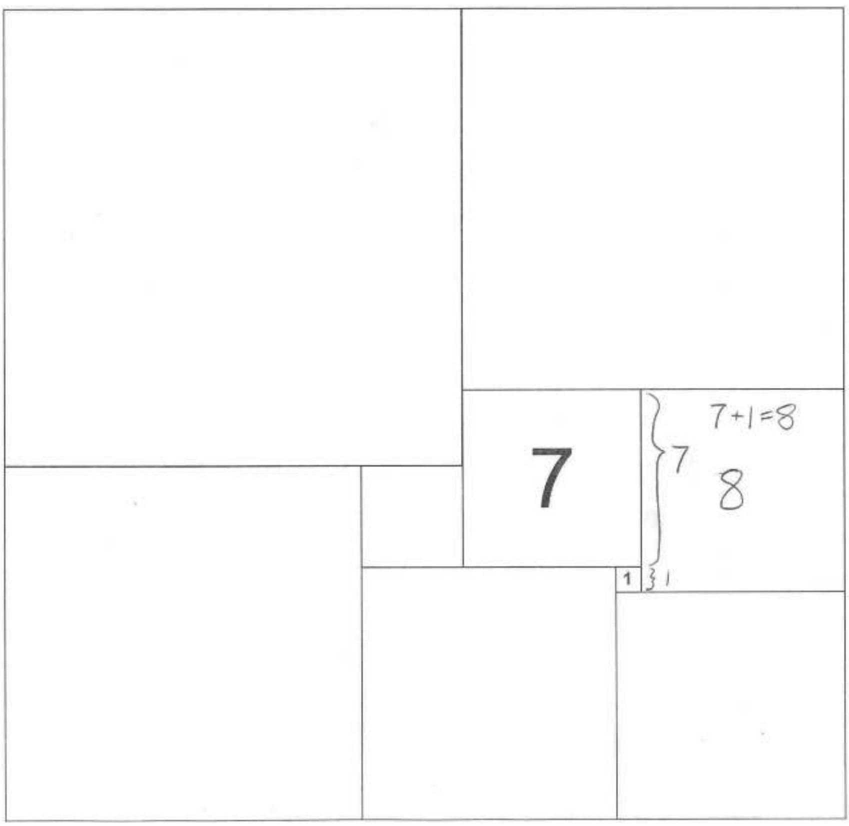
Find this and other Math for Love puzzles online at mathforlove.com/lesson-plan/.

Squaring Puzzles

by Gord Hamilton, Math Pickle

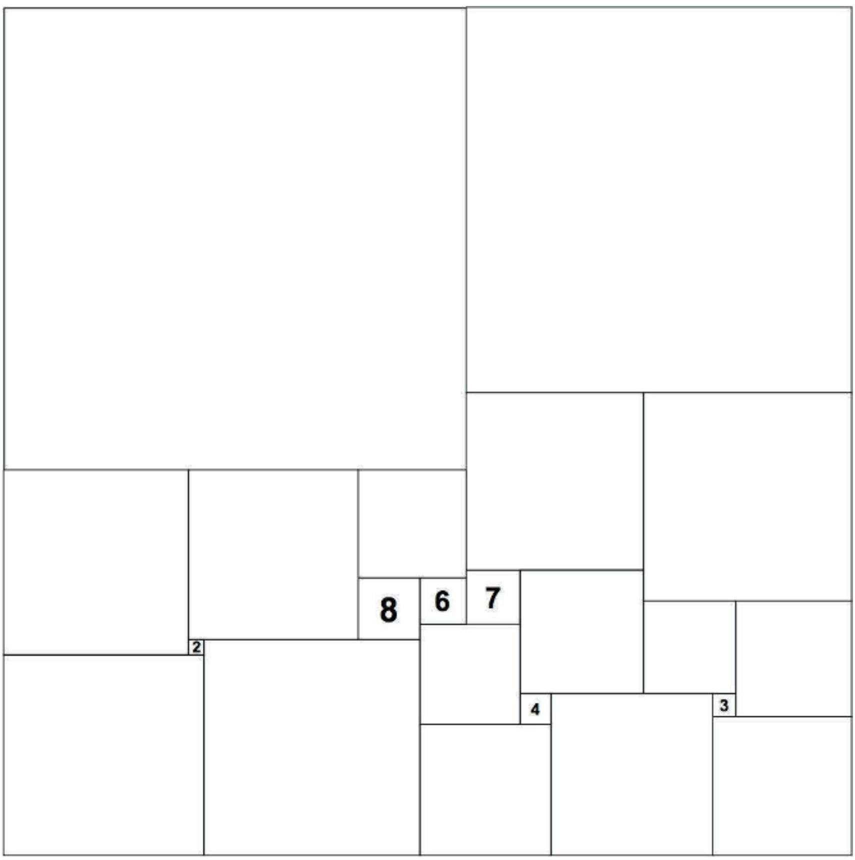
These abstract squaring puzzles give students addition and subtraction practice with numbers usually below 100. They also link these numerical activities to geometry. What a beautiful way to practice subtraction! *—Gord Hamilton, Founder of Math Pickle.*

The number in each square represents the length of a side of that square. Determine the length of a side of all the squares in this rectangle and the lengths of the sides of the rectangle.



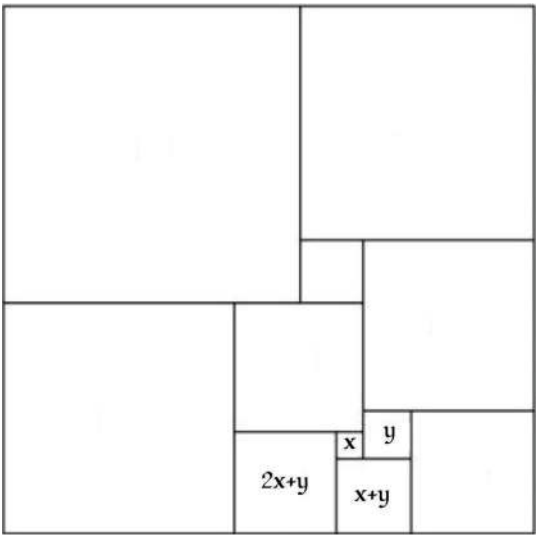
Find more square and subtracting puzzles here:
mathpickle.com/project/squaring-the-square/.

Here's a more challenging puzzle. As in the previous puzzle, the number in each square represents the length of the sides of that square. Determine the dimensions of all the squares in this rectangle and the lengths of the sides of the rectangle.



Algebra on Squares

by Gord Hamilton, Math Pickle
mathpickle.com/project/algebra-on-rectangles



Imagine all the interior rectangles are squares. The letter in each square represents the length of a side of that square.

Determine the length of a side of each square in this rectangle and write it inside the square.

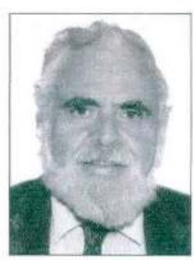
Also determine the lengths of the sides of the rectangle.

Find more of these algebra puzzles on the MathPickle link above.

If you want even more of a challenge, try the following puzzle.

Golomb's Puzzle Column™ Number 35: Rectangles With Consecutive-Integer Sides

Solomon W. Golomb



The sides (lengths and widths) of five rectangles measure each of the values 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 (units), in an unspecified order. As one of many possibilities, the five rectangles could be 1×6, 2×3, 4×10, 5×8, and 7×9, in which case their total area would be $6 + 6 + 40 + 40 + 63 = 155$ (square units).

1. How many different sets of five rectangles are possible? (The sequential order of the five rectangles does not matter, and we do not distinguish between an $a \times b$ and a $b \times a$ rectangle.)

2. What are the maximum and minimum values (A_{max} and A_{min}) for the total areas of the five rectangles?

3. Between A_{min} and A_{max}, which integer values of total areas are possible, and which are impossible?

4. There are a few sets of five rectangles (of the type we are considering) which can be assembled (without gaps or overlaps) to form a square.
   a.) Can you show, by a simple argument, that the total number of such sets of rectangles must be *even*?
   b.) Can you show that the side of any square so formed must have an *odd* length?

5. Can you exhibit any or all of the sets of rectangles, and the squares they form, as described in Problem 4?

Reproduced with permission of puzzle originator Solomon W. Golomb.

September 1996
IEEE Information Theory Society Newsletter

Trapezoidal Numbers

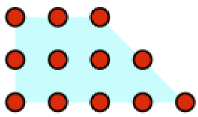
Compute

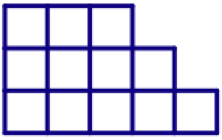
1. What is the sum 3 + 4 + 5?

2. What is the sum 4 + 5 + 6 + 7 + 8?

3. What is the sum 5 + 6 + ... + 80 + 81?

All of the results of these computations are called *trapezoidal* numbers, because you can draw a trapezoid that illustrates the answer to problem 1 with dots or blocks like this:





where each row has one more dot than the row before. So for instance 13 is trapezoidal because it is equal to 6 + 7. A trapezoidal number has to have at least two rows.

Patterns

4. What numbers can be written as 2-row trapezoidal numbers, like 13?

5. What numbers can be written as 3-row trapezoidal numbers, like 3 + 4 + 5?

6. What numbers can be written as 4-row trapezoidal numbers?

7. What about 5-row, 6-row, and so on? Can you explain a general rule, so that we can tell whether 192 is a 12-row trapezoidal number?

8. Can you name a large number that is not trapezoidal, no matter what number of rows you try? How do you know it can't be trapezoidal?

9. Can you name a large number that is trapezoidal in only one way? How do you know?

10. How many trapezoidal representations does 100 have? Why? How about 1000?

11. How many trapezoidal representations does 221 have? Why?

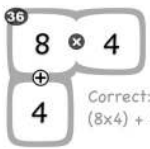
12. How can you determine how many trapezoidal representations a number has?

13. What if we allow negative numbers, like –2 + –1 + 0 + 1 + 2 + 3 + 4 + 5, in a trapezoidal representation? What if we allow “staircases” like 3 + 7 + 11?

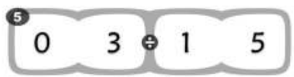
Find more Julia Robinson Mathematics Festival problem sets at jrmf.org/problems.php.

Cartouche puzzles

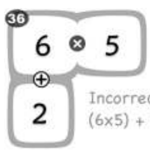
In each of these puzzles choose four digits from 0 through 9. Place one of these in each row and column. Use the operators in the white regions first, then between white regions. If there is no operator, the digits form a single number. The operator ⊖ takes the difference of the two numbers. The operator ÷ divides the bigger number by the smaller number.



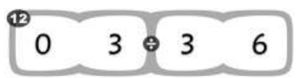
Correct:
(8x4) + 4 = 36



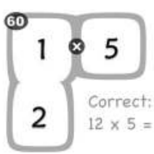
Correct:
15 ÷ 3 = 5



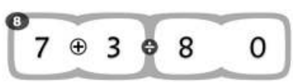
Incorrect:
(6x5) + 2 ≠ 36



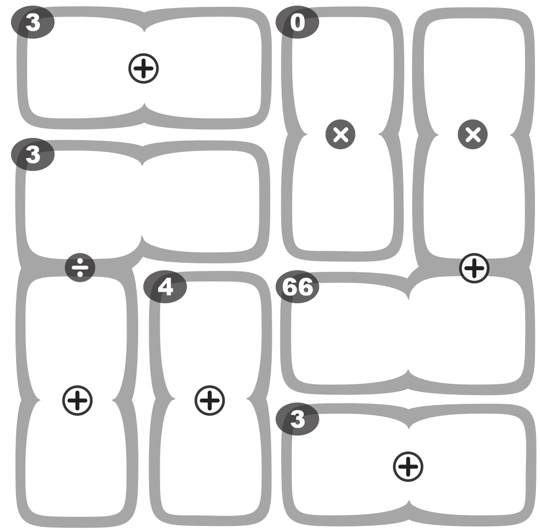
Incorrect:
36 ÷ 3 = 12
but the row contains duplicate 3s.



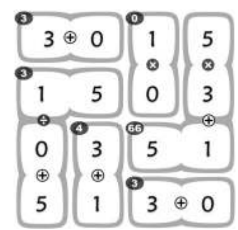
Correct:
12 x 5 = 60



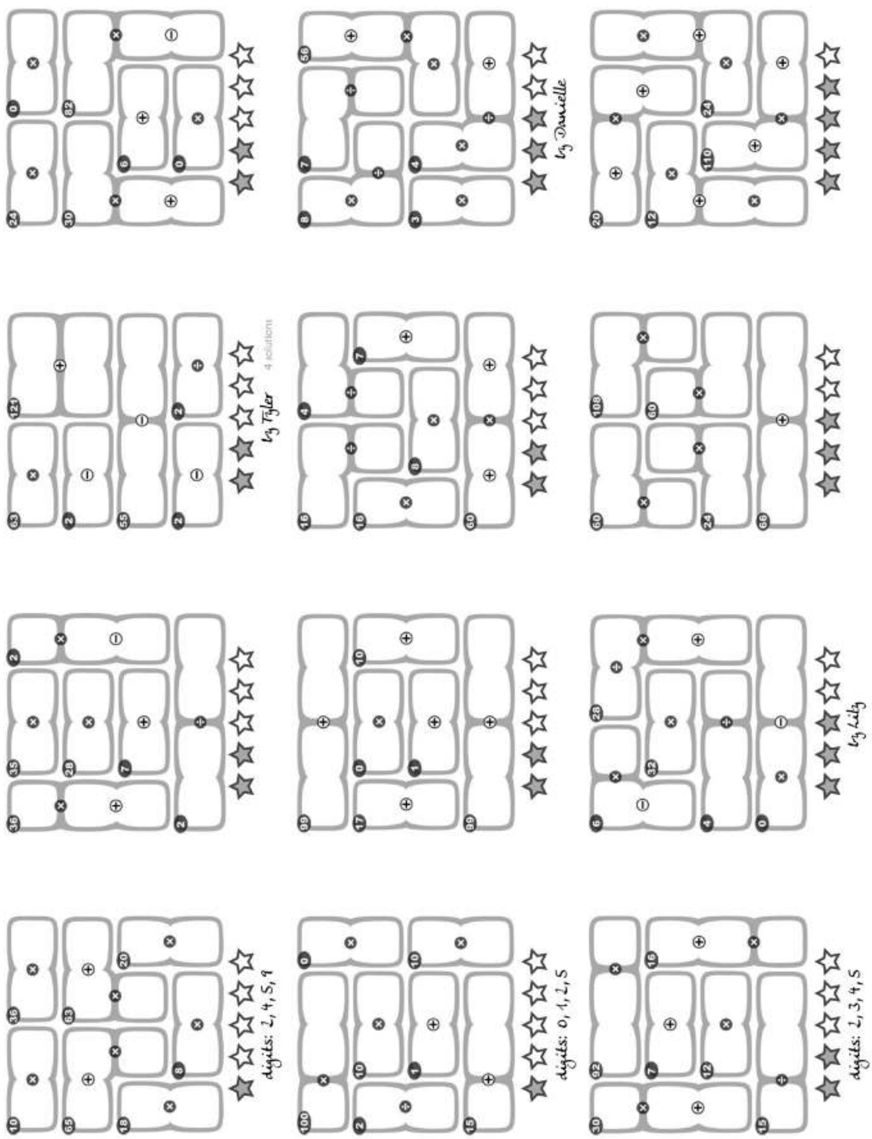
Correct:
80 ÷ (7+3) = 8



Sample Puzzle



Solution

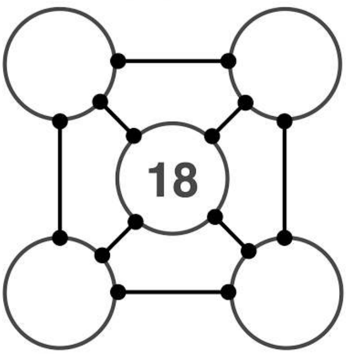


Find more MathPickle Cartouche puzzles online at mathpickle.com/project/cartouche/.

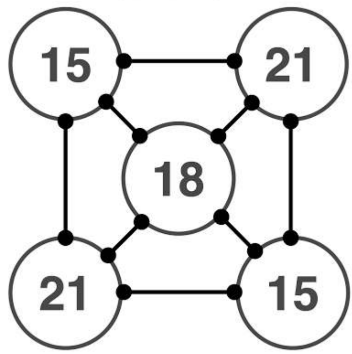
Digit Sums and Graphs

In each diagram, fill in the circle with positive whole numbers in such a way that each circle's number is the sum of the digits of all the numbers connected to it. Thanks to Erich Friedman for this idea!

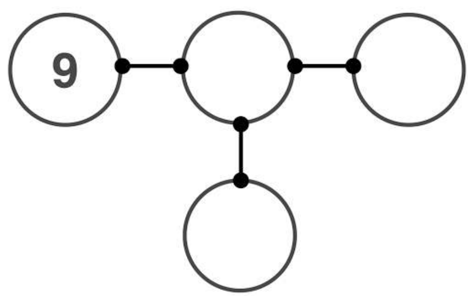
EXAMPLE

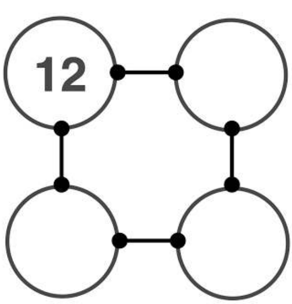


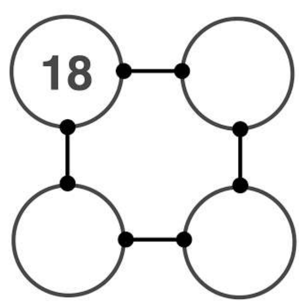
SOLUTION



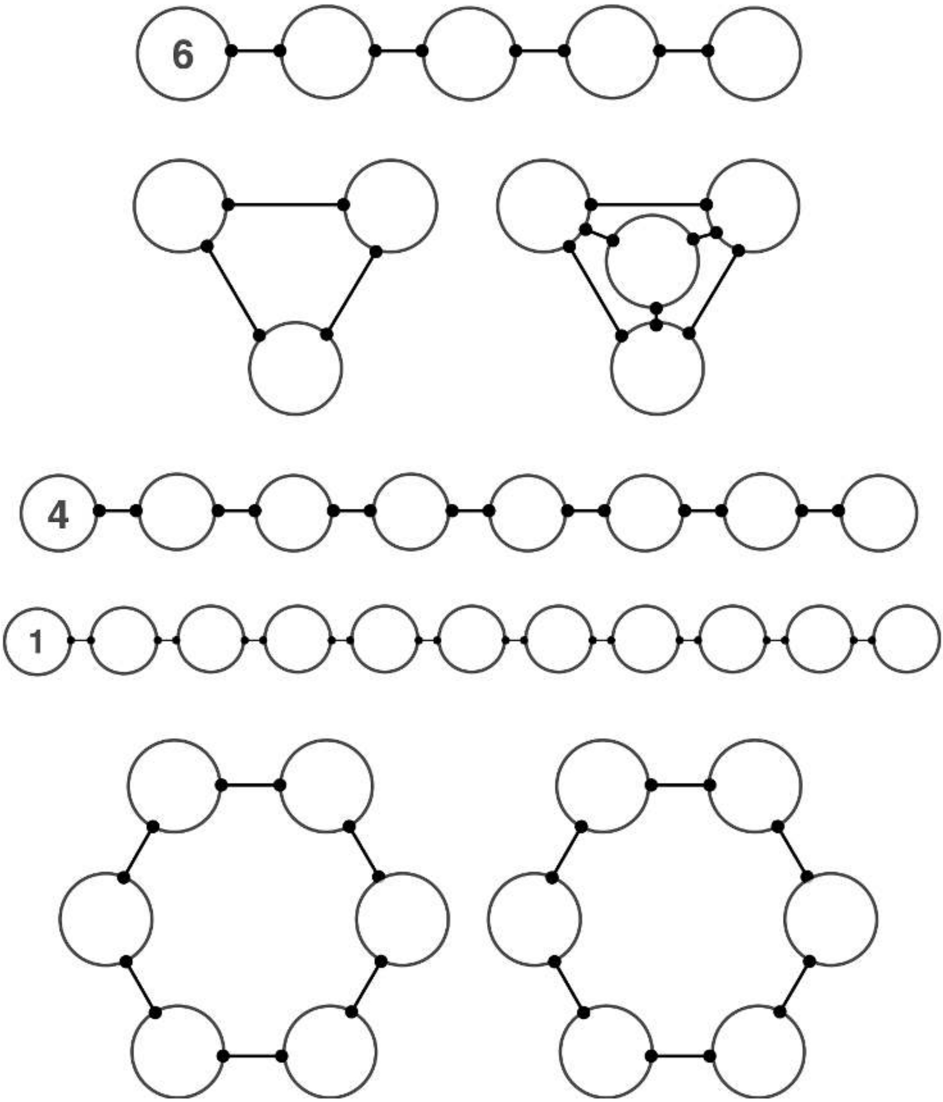
The solution works because
15 = (2+1) + (1+8) + (2+1) for the two corners
21 = (1+5) + (1+8) + (1+5) for the other two corners
18 = (1+5) + (2+1) + (1+5) + (2+1) in the center







Some of these may have more than one solution



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Switching Light Bulbs

A long hallway has 1000 light bulbs with pull strings, numbered 1 through 1000. If the light bulb is on, then pulling the string will turn it off. If the light bulb is off, then pulling the string will turn it on. Initially, all the bulbs are off.

At one end of the hallway, 1000 people numbered 1 through 1000 wait. Each person, when they walk down the hallway, will pull the string of every light bulb whose number is a multiple of theirs. So, for example, person 1 will pull every string; person 2 will pull the strings of bulb number 2, 4, 6, 8, 10, …, and person 17 will pull the strings of bulb number 17, 34, 51, 68, … .

For each situation below, which light bulbs are on after all the indicated people are done walking?

1. Everyone
2. The evens, or in other words, all the people whose numbers are even.
3. The odds
4. The primes
5. The perfect squares
6. The multiples of 3
7. The perfect cubes
8. The people 1 more than a multiple of 4.
9. The people 2 more than a multiple of 4 (that is, the evens not divisible by 4).
10. Any other interesting sets you'd like to consider?
11. Given the set of people who walked, what is a general strategy for figuring out which light bulbs are turned on?

For each situation below, which people should walk in order for the indicated sets of light bulbs to end up being the only ones turned on?

12. All the bulbs.

13. The odds, or in other words, all the light bulbs whose numbers are odd.

14. The evens

15. The primes

16. The perfect squares

17. The perfect cubes

18. The multiples of 3

19. The multiples of 4

20. The multiples of 6

21. Any other interesting sets you'd like to consider?

22. Given the set of light bulbs that are turned on, what is a general strategy for figuring out which people walked?

23. For any set of light bulbs, does there necessarily exist a set of people who can walk such that the given set of light bulbs ends up being the only set turned on? If so, prove it. If not, describe the sets of light bulbs that are impossible.

24. Suppose that there are still 1000 people, but there are more than 1000 light bulbs. Not knowing which people walked, but only knowing which of the first 1000 light bulbs are turned on, what can you predict about which of the bulbs beyond #1000 are turned on?

Thanks to Stan Wagon's Macalester problem of the week for the idea behind this extension of the famous locker problem. Thanks to Glenn Trewitt and Car Talk for the idea of using light bulbs instead of lockers.

Find more Julia Robinson Mathematics Festival problem sets at jrmf.org/problems.php.

Casting Out Nines

The "digital root" of a number is the result you get if you add up its digits, and then add up the digits of that result, and so on, until you end up with a single digit. For instance, the digital root of 44689 is computed by finding that $4 + 4 + 6 + 8 + 9 = 31$, and then $3 + 1 = 4$ gives you a single-digit answer.

1. Let's look at two numbers that add up to 44689, such as 31847 and 12842. What relationship can you find among the digital roots of these numbers?

2. What about two numbers that subtract to make 44689, like 83491 and 38802? Is there a relationship among their digital roots? What can you do with 100000 and 55311?

3. What about two numbers that multiply to make 44689, like 67 and 667? Or two other numbers that multiply to make 44689, like 23 and 1943?

4. The process of taking the digital root is called "Casting out nines" for a reason: what you're actually doing in computing the digital root is another way of determining the remainder when you divide by 9. In other words, you keep throwing away multiples of 9 until you're eventually left with a number smaller than 9. Well, that's not quite true: why not?

5. In the original example of 44689, we obtained 31 after the first step. Let's see the 9s disappearing as we go from 31 to 3 + 1: 31 means $3 \times 10 + 1$ which is the same as $3 \times 9 + 3 \times 1 + 1$, so after throwing away the 9s we have $3 \times 1 + 1$, which finally is $3 + 1$. Can you give a similar explanation for how 44689 turns into $4 + 4 + 6 + 8 + 9$ after throwing away a lot of 9s?

6. One of the major uses of casting out nines is to check arithmetic quickly. If your calculation (like in the first few problems here) doesn't match up, then you know there was an arithmetic mistake. Which of the following can be proved wrong by casting out nines? Are the other ones actually correct?
   a) $1234 + 5678 = 6812$
   b) $12345 - 9876 = 2469$
   c) $10101 - 2468 = 7623$
   d) $1234 \times 5678 = 7006652$
   e) $4321 \times 8765 = 37783565$
   f) $345 \times 543 = 196335$
   g) $2^{17} = 130072$ (warning! How should you handle exponents?
   Think about this very carefully!)

7. On the other hand, certain kinds of mistakes will never be found by casting out nines. Can you give some examples of these? Examples that might be common?

8. Why is this process a bad idea for division when it works so well for addition, subtraction, and multiplication? Give an example where casting out nines seems to be "wrong" even though the answer is correct.

9. On the other hand, you can use casting out nines to check division problems by rewriting them as multiplication and addition. How would you rewrite "23894 divided by 82 is 291 with a remainder of 32" using only multiplication and addition, so you could then check it by casting out nines?

10. Another way to think about casting out nines is that as you add 9 to a number, you increase the tens digit by 1, and decrease the ones digit by 1, so adding 9 won't change the digital root. What is the flaw in this logic? Can you repair it?

11. Casting out nines has some other interesting applications as well. What is the digital root of 3726125? Can you use that information to explain why 3726125 is not a perfect square?

12. You can also cast out elevens instead of nines. Start with the rightmost digit, and alternately add and subtract. So with 44689 you'd take $9 - 8 + 6 - 4 + 4 = 7$. If you end up with a negative number, remember you're casting out elevens, so just add 11 as many times as you'd like. Can you explain why this process works?

13. There are some common mistakes that you wouldn't be able to catch with casting out nines, but you can catch by using casting out elevens. Give at least one example.

14. There's a magic trick that is most often done using a calculator. Pass the calculator around the room, and each person types in one digit and presses the multiplication key. After a while, the calculator screen is full of digits. The person holding the calculator at that point eliminates any one digit 1 through 9 (not 0), and then takes the remaining digits and writes them in any order. For example, they might write 3004129. Then, a mathematician almost instantly says what the missing digit is. Which digit is missing? How could the mathematician know? But sometimes the mathematician is wrong. Why?

15. What is the digital root of 4444^{4444}? Can you determine how many times you will have to sum the digits before obtaining a single digit answer?

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MathPickle

Gord Hamilton has a passion for getting students to realize that mathematics is beautiful. (Grades K-12)
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galileo.org

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galileo.org/classroom-examples/math/math-fair-problems

The New York Times — NUMBERPLAY

On the NY Times website, Numberplay generally presents mathematical and/or logical puzzles and problems. (Grades 5-Adult)
wordplay.blogs.nytimes.com/category/Numberplay

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MathsChallenge.net

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Project Euler.net

Project Euler offers for free engaging computation problems that will require more than just mathematical insights to solve. (Grades 5-Adult) projecteuler.net

Math Central is an award-winning website with investigations for teachers and students. (Grades 7-12)
mathcentral.uregina.ca/mp

Gathering 4 Gardner

G4G features puzzles, games, magic tricks, and crafts. (Grades K-Adult)
celebrationofmind.org/puzzles_games

Dan Meyer has created problems and videos to inspire students to solve problems. (Grades 4-12)
blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story

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