A group activity to build an aperiodic spectre tiling

Scott Vorthmann[1], Shiying Dong[2], and Chaim Goodman-Strauss[3]

[1]Saratoga, CA; scott@vorthmann.org
[2]Greenwich, CT; shiyingdong@gmail.com
[3]National Museum of Mathematics, New York, NY; chaimgoodmanstrauss@gmail.com

Abstract

This paper describes a procedure for a group activity to build a large patch of aperiodic spectre tiles, using individually cut paper tiles. The activity lets participants understand the substitution tiling rule.

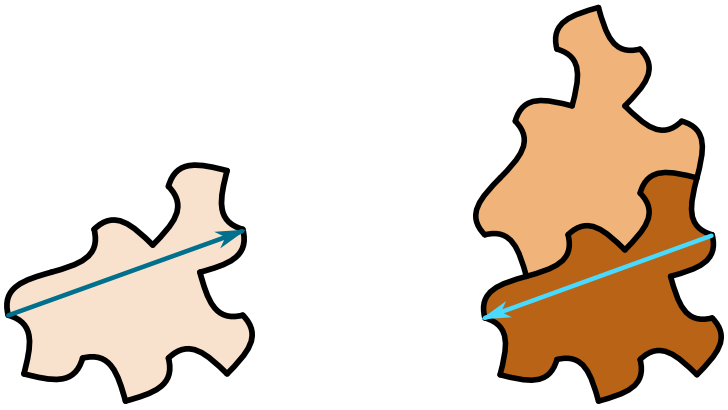
Last year's discoveries of the hat ([2]) and spectre ([3]) aperiodic monotiles were tremendously exciting. Anyone could run Craig Kaplan's wonderful web application ([1]) and generate a large swatch of aperiodic spectre tiling. However, in doing so, we all missed an opportunity to build intuition about how the substitution tiling actually works.

This paper describes a fairly simple group activity to manually perform the spectre substitution tiling, using nothing but paper tiles and tape. The activity is simple enough for children, and is limited only by how much space you have and how many tiles you can cut out.

A version of these instructions is available online ([4]) with more figures and more detailed instructions.

1. Prepare the space and materials. This activity is designed to require 488 separately cut paper spectre tiles. Realistically, this implies some kind of automated cutting. We used an Accucut die cutter, and dies created by Custom Shape Pros ([5]). Our spectre template is available on request.

With one tile filling a standard letter-size sheet of paper, the complete assembly will require roughly 20 feet by 16 feet of area... but perform your own calculation using our figures!



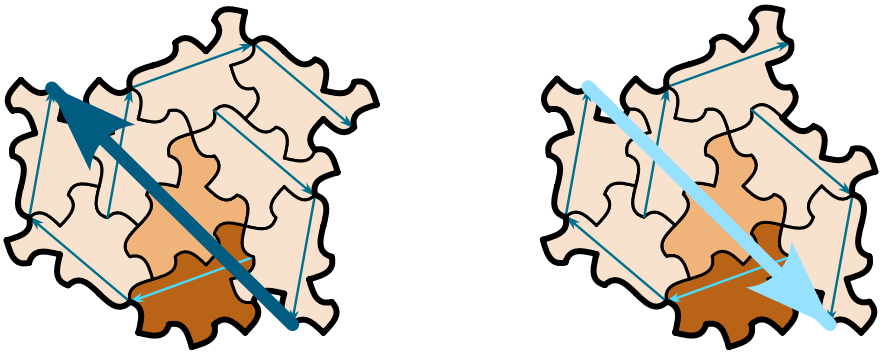
(a) *spectre* (b) *mystic*

Figure 1: *Marked spectre and mystic tiles.*

2. Assemble the spectre-2 supertile. Start by marking arrows on 7 spectre tiles as shown in Figure 1(a), using a light pencil mark. Take two *additional* tiles and assemble the mystic, then mark the bottom tile with

an arrow *in the opposite direction*, as shown in Figure 1(b). You may want to use a distinct paper color (or two) for your mystic assemblies, as we do in our figures.

Now, lay out your 7 marked spectre tiles around the mystic as shown in Figure 2(a). A simple way to do this is to start from the mystic and add two spectre tiles to the left, such that the arrows form half of a regular hexagon, with arrows joined head-to-tail. Now add another spectre "inside" the half-hexagon, with its arrow parallel to the last arrow, and add two more to form another half-hexagon. Finally, insert a spectre again inside the last hexagon leg and parallel to it, and add one more spectre to form just one corner of a hexagon. Notice that all three hexagon fragments have a clockwise orientation as you follow the arrows around.

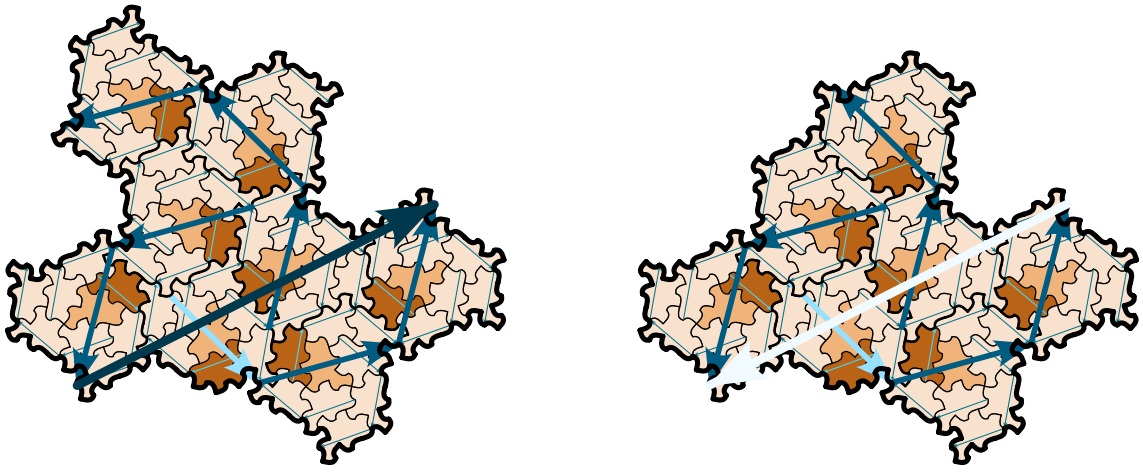


(a) *spectre-2* (b) *mystic-2*

Figure 2: *Marked spectre-2 and mystic-2 supertiles.*

Apply scotch tape to each place where two spectre tiles meet in this assembly. Add a line of painter's or masking tape joining the head of the first half-hexagon to the head of the last hexagon fragment, including an arrowhead. We'll call this your "spectre-2" template. Now that you have this template, you can copy it 47 times, without the pencil marks this time. You do need to add the final tape arrow to each spectre-2 supertile, since these marks will guide your construction of the larger supertiles.

All of the figures in the document are aligned such that you can always find the original mystic at the bottom center. We recommend that you do the same with your physical tile assemblies... orientation is the trickiest thing to manage in this procedure!



(a) *spectre-3* (b) *mystic-3*

Figure 3: *Marked spectre-3 and mystic-3 supertiles.*

3. Assemble the mystic-2 supertile. The mystic-2 supertile is shown in Figure 2(b). It is very similar to the spectre-2, but it omits one spectre tile on the upper right, and the *tape arrow has the arrowhead on the opposite end.* Use your spectre-2 template to build a mystic-2 template, being careful to orient the tape arrow correctly. Create 6 more mystic-2 supertiles using this template.

4. Assemble the level-3 supertiles. Creating the level-3 supertiles is almost identical to creating the spectre-2 and mystic-2, except *right and left are reversed, and the arrows flow counter-clockwise*. See Figure 3(a) for the spectre-3 layout. Start with a mystic-2, and add two spectre-2 tiles so that the three head-to-tail arrows form a half-hexagon *to the right*, circling counter-clockwise. Continue to mirror the level-2 pattern in this way. Scotch-tape the supertiles together at the edges so you can move them around in the next level, and add the large tape arrow as indicated, being careful where the arrowhead goes. Build 5 more of these spectre-3 supertiles.

The mystic-3 supertile is shown in Figure 3(b). As before, it is identical to the spectre-3 supertile, except for a missing spectre-2, *this time on the upper left*, and again a reversed final tape arrow. You only need one mystic-3; be careful to get the arrowhead on the correct end.

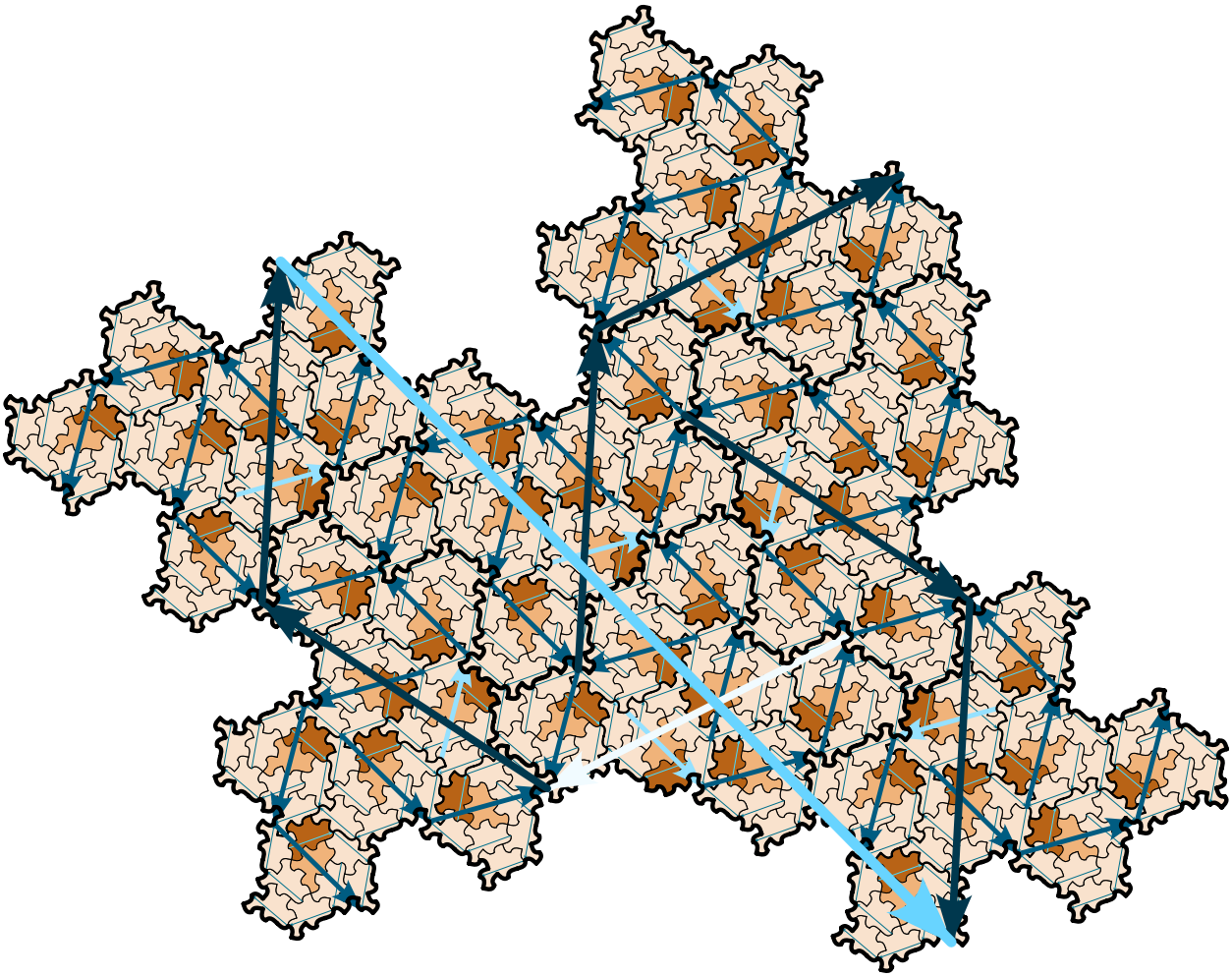


Figure 4: *Marked mystic-4 supertile.*

5. Assemble the mystic-4. The final step in these instructions produces a mystic-4 supertile (shown in Figure 4), just because it requires fewer original tiles than the spectre-4. You are, of course, free to construct a spectre-4 instead, or even to build enough of them to assemble a spectre-5 or mystic-5, with thousands of original spectre tiles!

The procedure again is the same as the prior two steps, with right and left reversed *again* with respect to the level-3 steps, and arrows now flowing *clockwise* again. The mystic-4 starts with your mystic-3 at the bottom center, and you add spectre-3 supertiles to the left, with arrow paths curling around clockwise.

This pattern of reversal at every level of substitution continues indefinitely, of course. Now you have the tools to assemble an aperiodic tiling of arbitrary size; find a gymnasium, organize a crowd of participants, and set a world record!

Conclusions

The procedures described above were designed for a recent group build activity at the 15th Gathering for Gardner conference, and were further refined during and after that activity. Since the process became simpler as our understanding increased through experience, we thought it important to capture and share it. Hopefully these instructions will unlock similar activities in schools worldwide, fostering the next generation of tiling enthusiasts and mathematical creators.

References

[1] C. Kaplan. *Spectre tile generator*. https://cs.uwaterloo.ca/~csk/spectre/app.html

[2] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. "An aperiodic monotile." 2023. https://arxiv.org/abs/2303.10798

[3] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. "A chiral aperiodic monotile." 2023. https://arxiv.org/abs/2305.17743

[4] S. Vorthmann. Spectre tiling activity instructions. https://www.gathering4gardner.org/spectre-tiling-activity

[5] *Custom Shape Pros service*. https://www.customshapepros.com/