Gathering 4 Gardner 14

G4G14 Exchange Book

Art, Games, Magic, Mathematics, Puzzles, Legacy, & Science

Atlanta, Georgia

APRIL 6 - APRIL 10, 2022

G4G14 Exchange Book

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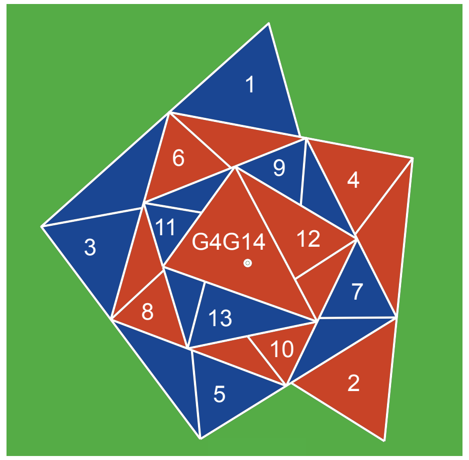
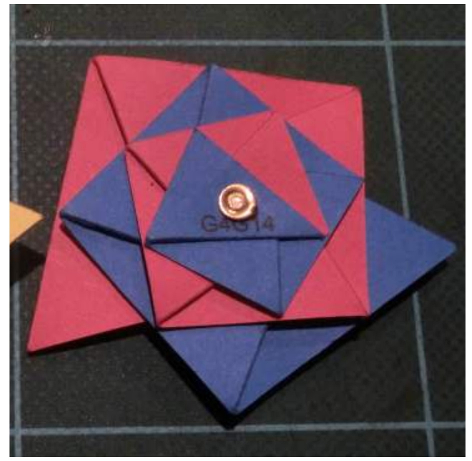
LEGACY

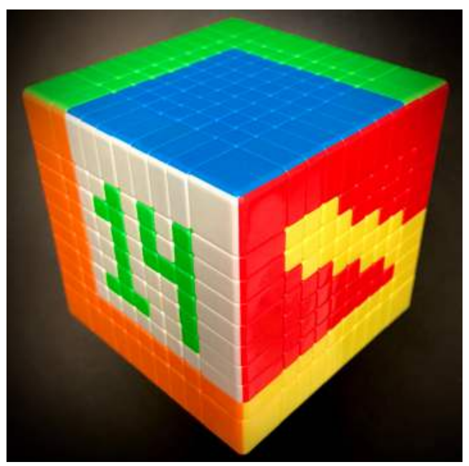
SCIENCE

Unofficial Logos

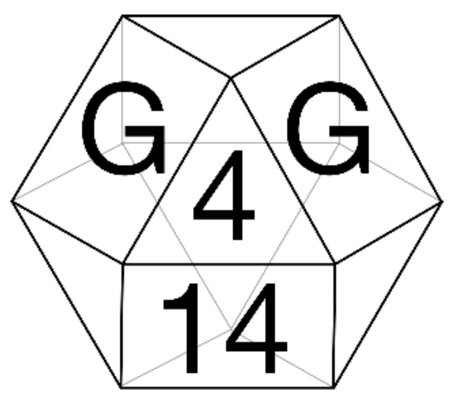




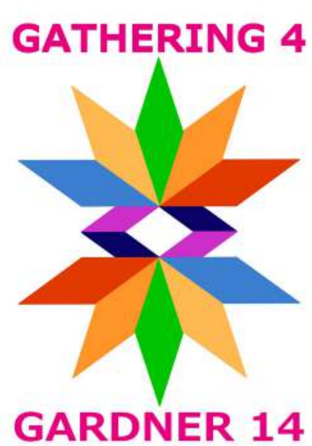
Submitted by
Akio Hizume



Submitted by
David Plaxco



Submitted by
James Propp



Submitted by
Matjuska Teja Krasek

Gathering 4 Gardner 14
Presentation Schedule

Thursday, April 7th, 2022

FEATURED PRESENTATION

(*) = VIRTUAL PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

	Speaker	Title	
8:30 AM	Skona Brittain	14 Numbers for G4G14	*6 min*
	Kenneth Brecher	The Sirius Enigmas Mathematical Tops	*6 min*
	George I. Bell	Fun with tops	*6 min*
	Sabine Segre	Rep-Tile Tangram	*6 min*
	Ryuhei Uehara *	Solving Rep-tile by Computers	*6 min*
	Robert W Vallin	An Unexpected Appearance (or Look Ma, No Rectangles)	*6 min*
	Jorge Nuno Silva *	Magic and problems from half a millennium ago	*6 min*
	Rick Sommer	Knuth's Up-arrow into the Transfinite, and Beyond!	*6 min*
	Nathaniel Segal	Performing On The Virtual Stage	*6 min*
	Adam Atkinson *	Daedalus	*6 min*
	Raymond Hall	@physicsfun: Social Media as a Museum of Science and Math (and A Recent Path Down a Rabbit Hole with Pentomino Tilings)	*6 min*
10:00 AM		*Break*	*30 min*
10:30 AM	Ingrid Daubechies	Mathemalchemy	*45 min*
	Darren Glass	Chutes and Ladders with Multiple Dice	*6 min*
	Peter Cannon	Interviewing Martin Gardner	*6 min*
	Hector Rosario *	Playing in Gardner's Wonderland	*6 min*
	Alexandre Owen Muñiz	Which Pentomino Is the Least Convex?	*6 min*
	Erika Roldan	Guarding Art Galleries with Rooks and Queens	*6 min*
12:00 PM			

Afternoon Session: 1:30 PM - 5:15 PM

	Speaker	Title	
1:30 PM	Various	Tributes to Berlekamp, Conway, Guy, Graham, and Randi	*60 min*
	Ryan Hayward *	Hex: Two Books and One Puzzle	*6 min*
	Jonathan Bobrow	Open-Source Physical Cellular Automata	*6 min*
	Robert Bosch	Domino Steganography and Lenticular Dice Mosaics: Two Examples of Opt Art	*6 min*
	George Hart	Warped-Grid Jig-Saw Puzzles	*6 min*
	Peter Knoppers	2e9s - An Egocentric View of Time	*6 min*
	Kate Jones *	StarHex-14, The beauty of polyform sets	*6 min*
	Gabriel Kanarek	A Variation on the Erdős-Straus Conjecture	*6 min*
3:30 PM		*Break*	*30 min*
4:00 PM	Red Deupree	Tetraflexagons / Coloring the Penrose Pattern	*6 min*
	Ann Schwartz	I am the Rhombus, Goo Goo G'Joob. Introducing the Rhombus Flexagon!	*6 min*
	Michael Keith *	300+ Digits Of Pi From An (Almost) Ordinary Deck Of Cards	*6 min*
	Daniel Kline	Playing with Puzzles: A Sample of What's New at the Julia Robinson Mathematics Festival	*6 min*
	Koji Fujimoto *	The Actual 26 Integers for a Diophantine Representation Those Make Prime Number 2.	*6 min*
	Alexa Meade	Adventures in Wonderland	*6 min*
	Chaim Goodman-Strauss	Tooti Tooti!	*6 min*
5:15 PM			

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Gathering 4 Gardner 14
Presentation Schedule

Friday, April 8th, 2022

FEATURED PRESENTATION

(*) = VIRTUAL PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

	Speaker	Title		
8:30 AM	Jason Rosenhouse	The Use and Abuse of Probability in Evolutionary Biology	*6 min*	8:30 AM
	Robert P Crease	From MG to QB	*6 min*	
	Carolyn Yackel	Using Mathematics to Inform Fiber Arts Work	*6 min*	
	Andrew Rhoda	The Slocum Mechanical Puzzle Collection at the Lilly Library	*6 min*	
	Steven Landsburg	Why Do People Stand Still on Escalators But Not on Stairs?	*6 min*	
	Alba Marina Málaga Sabogal *	Paper Tori	*6 min*	
	Barry Cipra	A Toroidal Looping Puzzle	*6 min*	
	Barry Hayes *	Is the Szilassi Polyhedron Unique?	*6 min*	
	John Edward Miller	More Fun with Langford's Problem	*6 min*	
10:00 AM		*Break*	*30 min*	10:00 AM
10:30 AM	Mark Burstein	A Literary Englishman and the Scientific American: Lewis Carroll's Appearances in 'Mathematical Games'	*45 min*	10:30 AM
	Stanley S Isaacs	Lewis Carroll, Mathematician Rediscovered: Euclid and Non-Euclidean Geometry	*6 min*	
	Stuart Moskowitz	Lewis Carroll, Mathematician Rediscovered: Trigonometry, Recreational Math, Logic, and More	*6 min*	
	Chris Staecker	Gerber's Great Graphical Gizmos	*6 min*	
	Hideki Tsuiki *	Imaginary Cube Puzzles in classes	*6 min*	
	T. Arthur Terlep	Waving Goodbye to Berlekamp	*6 min*	
12:00 PM				12:00 PM

Afternoon Session: 1:30 PM - 5:30 PM

	Speaker	Title		
1:30 PM	Robert P Crease	Fourteen: Workhorse Wondr	*6 min*	1:30 PM
	Delicia Kamins *	Beware You All, Something Fourteen This Way Comes	*6 min*	
	Dana S Richards	"Are you a mathematician?"	*25 min*	
	R. William Gosper	The Dragon Function is way cooler than the "Dragon Curve".	*8 min*	
	Adam Rubin	Further Abracadoodads	*6 min*	
	Bjoern Muetzel *	Mirror Solids	*6 min*	
	Eleftherios Pavlides	Tetradecahedron as Palimpsest of the Monododecahedral 1-Parameter Family of the Pavlides Elastegrity	*6 min*	
	Lauren Siegel	Making Math	*6 min*	
	Peter Winkler *	Two Paradoxes of Slight Bias	*6 min*	
	Barney Sperlin	The Accountant	*6 min*	
	Spandan Bandyopadhyay *	Outsider Mathematics	*6 min*	
	Bill Ames	Snowflake Formation	*6 min*	
3:30 PM		*Break*	*30 min*	3:30 PM
4:00 PM	Joe Buhler	Wildly Nontransitive Dice	*6 min*	4:00 PM
	Susan Goldstine	Mosaic Knitting Friezes: seventeen symmetries, minus three	*6 min*	
	Philipp Legner *	Mathigon – Textbook of the Future	*6 min*	
	Ricardo Teixeira	Magical Journey through Advanced Mathematics	*6 min*	
	Miquel Duran *	Quantum Science and Card Magic: from basic concepts to cryptography	*6 min*	
	James Gardner	Growing up with Martin Gardner: Some Old and New Reflections	*30 min*	
5:30 PM				5:30 PM

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Gathering 4 Gardner 14
Presentation Schedule

Saturday, April 9th, 2022

(*) = VIRTUAL PRESENTATION

Morning Session: 8:30 AM - 11:15 AM

	Speaker	Title		
8:30 AM	Tomas Rokicki *	Twizzle: Twisty Puzzle Simulator in JavaScript	*6 min*	8:30 AM
	Lucas Garron *	Browsers, Bluetooth, and VR: From Physical to Virtual Twisty Puzzles	*6 min*	
	Carl Hoff	The Double Circle Real 6×6×6	*6 min*	
	David Plaxco	Knot Theory on the n*n*n Rubik's Cube	*6 min*	
	Robert Fathauer	Walkable Knots and Links	*6 min*	
	Karl Schaffer *	Dancing Topologically: Paths, Passes, and Puzzles	*6 min*	
	Chris Hibbert	Math in Ingress	*6 min*	
	Jeanine Meyer	Origami with Explanations	*6 min*	
	Akio Hizume *	Fibonacci Turbine	*6 min*	
9:45 AM		*Break*	*30 min*	9:45 AM
10:15 AM	Alexander Kernbaum	Seven Ways to Make an Ellipse — and One That Might be Useful	*6 min*	10:15 AM
	Aaron Siegel *	Adventures in 3D Puzzle Printing	*6 min*	
	Dave Buck	Playing Cards as Art	*6 min*	
	Nicole Dieker *	Music Performance	*4 min*	
	Scott Vorthmann	Customizing Zometool	*6 min*	
	George Hart	A Quick Summary of Hands-on Mathematical Activities Participants May Enjoy During the Excursion Afternoon	*10 min*	
11:15 AM				11:15 AM

Gathering 4 Gardner 14
Presentation Schedule

Sunday, April 10th, 2022

(*) = VIRTUAL PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

	Speaker	Title		
8:30 AM	Todd Wilk Estroff	The Ultimate Puzzle: a Psychiatrist Trying to Unravel Human Behavior	*7 min*	8:30 AM
	Dana Randall	Dumb Robots, Smart Algorithms	*6 min*	
	Lew Lefton	Mathematical Comedy	*6 min*	
	Sabetta Matsumoto	Mobius Cellular Automata Scarves	*6 min*	
	Delicia Kamins *	Fractal Top Down	*6 min*	
	Tiago Hirth	The First Recreational Mathematics Meeting	*6 min*	
	Roger Russell Manderscheid	The Amazing Number 153 and How it Captured Me	*6 min*	
	Cindy Lawrence	Million Millimeter March for MoMath	*6 min*	
	Jim Weinrich	Does Conway's "Game of Life" Predict That the Speed of Light is Constant?	*6 min*	
10:00 AM		*Break*	*30 min*	10:00 AM
10:30 AM	Bob Hearn	Rectangular Unfoldings of Polycubes	*6 min*	10:30 AM
	David Richeson	Circle Square Illusions	*6 min*	
	Yossi Elran	John Conway's Doomsday Rules for the Hebrew Calendar	*6 min*	
	Stephen Macknik	Champions from The Best Illusion of the Year Contest	*6 min*	
	Henry Segerman	Geared Mechanisms	*6 min*	
	Nancy Blachman	COVID Misinformation Spreads because Many Don't Understand Math	*6 min*	
	Erik Demaine *	New Adventures in Puzzle Fonts	*6 min*	
	P. Justin Kalef	The Whys (and Hows) of a Philosophical Teacher	*6 min*	
	Douglas McKenna	Half-Domino Curves in an Interactive Math Book	*6 min*	
	Cassondra Darling	Deciphering Wonderlands - Creating a Martin Gardner Digital Library	*6 min*	
12:00 PM				12:00 PM

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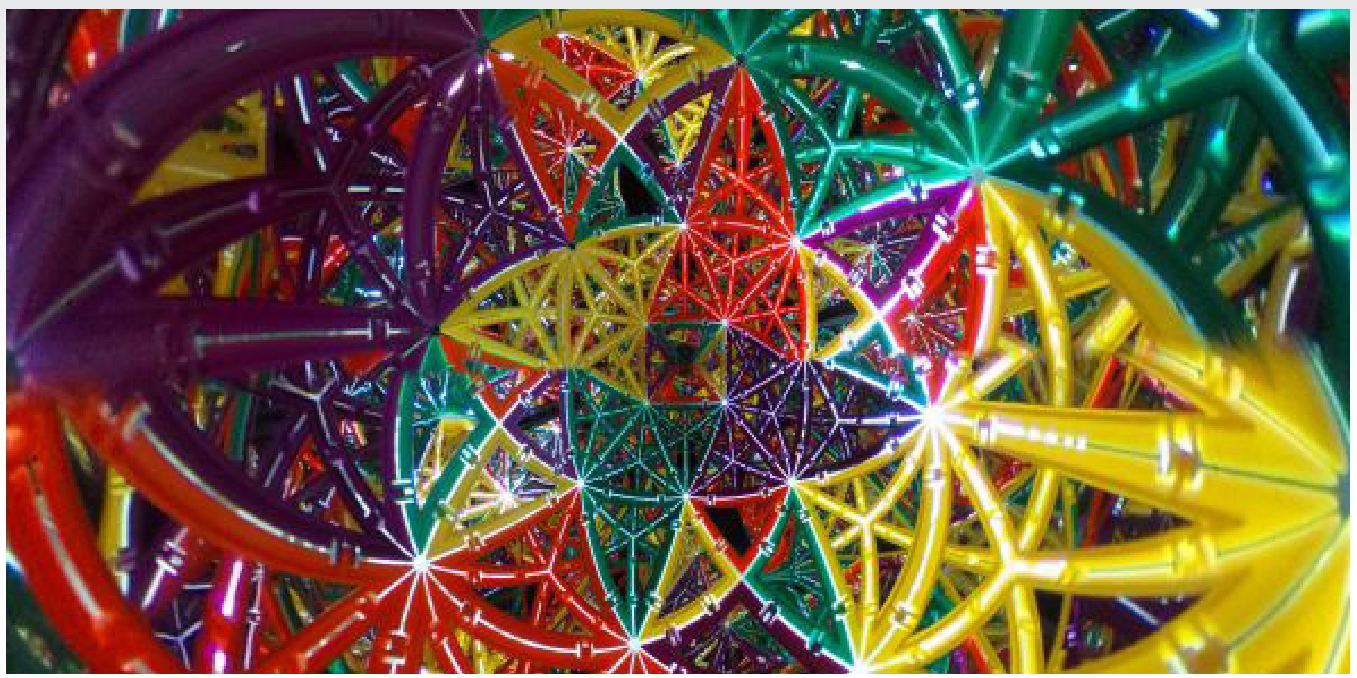
ART



Snub Cube Kaleidoscope | Bjoern Muetzel & Matthew Bajor | Page 26

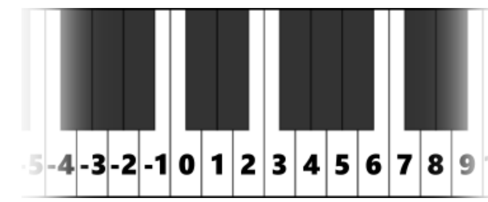
A Few Algorithms for Musical Harmonization

Neil Bickford

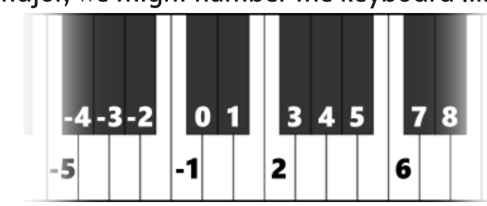
You can try out two of these algorithms online at https://neilbickford.com/G4GI4/index.htm!

Here's a neat way to generate a harmony from a melody.

Let's say we have a musical major scale — this could be C major, D-flat major, D major, or any other major scale. Number the notes in the scale consecutively, using the number 0 for the first scale degree in some octave. For instance, for C major, we might number the keyboard like this:



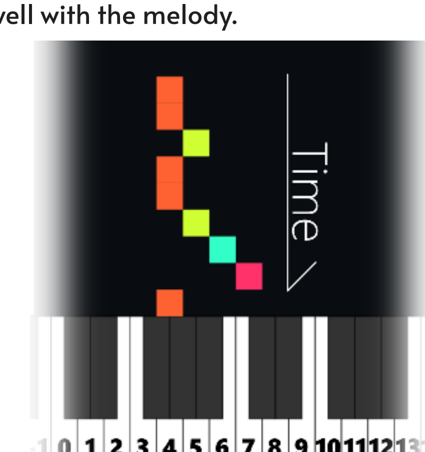
And for D-flat major, we might number the keyboard like this:



This makes it so notes with the same degree always have the same number taken mod 7.

This is a bit like the Nashville Number System, but here we've subtracted 1 from all numbers, and we have an unbounded range so we don't have to denote the octave separately.

Now, let's consider a melody using this numbering system. We'd like to generate three additional musical parts, using pitches below the melody, that harmonize well with the melody.



This shows the series of notes 4, 4, 5, 4, 4, 5, 6, 7, 4 in C major.

Traditionally, the four total parts are named the bass, tenor, alto, and soprano voices (as if we were writing for a SATB choir), counting from the lowest part to the highest part. However, this algorithm doesn't ensure that each part is within the typical singing part range, and it also doesn't always follow voice leading rules.

We'll start by creating three-note chords (triads) in three different positions, which we'll call position 0, position 1, and position 2.

This article's positions 0, 1, and 2 are also known as triads' root position, first position, and second position.

To create a position 0 triad, take the melody note *n* and add notes *n-2* and *n-4*. For instance, if our melody note was 9, we'd create the position 0 triad {5, 7, 9}. The *root note, r,* is the lowest note of the chord in position 0 (here, it's *r=n-4*).

To create a position 1 triad, start with the chord {n, n+2, n+4} (which is a position 0 triad with the melody note as the root). We want the melody note to be the highest part, so we'll *invert* the chord by taking all the notes above the melody and subtracting 7 to move them down an octave, then sort them from lowest to highest. This gives the chord {n-5, n-3, n}, with root note *r=n.*

Finally, to create a position 2 triad, follow the same procedure, but start by building a position 0 triad where *n* is the middle note. This gives the chord {n-5, n-2, n}, with root note *r=n-2.*

To add the bass part, add the root note shifted down an octave (*r-7*) —unless the root note is congruent to 6 mod 7! In that case, add *r-9.*

What's going on here is that when r=6 (mod 7), we get a tritone between the root and one of the other notes in the chord! Moving the root down two more notes turns this into an inversion of a dominant seventh chord, which usually sounds less dissonant to most listeners.

Another way of summarizing the above is that we'll generate one of three chords:

- `Position 0: {n-11, n-4, n-2, n}`
- `Position 1: {n-7,  n-5, n-3, n}`
- `Position 2: {n-9,  n-5, n-2, n}`

then if the lowest note is congruent to 6 mod 7, we subtract 2 from it.

To harmonize a melody, take each melody note in turn, generate the chord above for it, and then randomly choose one of the other two positions. Importantly, we never repeat the same position twice in a row!

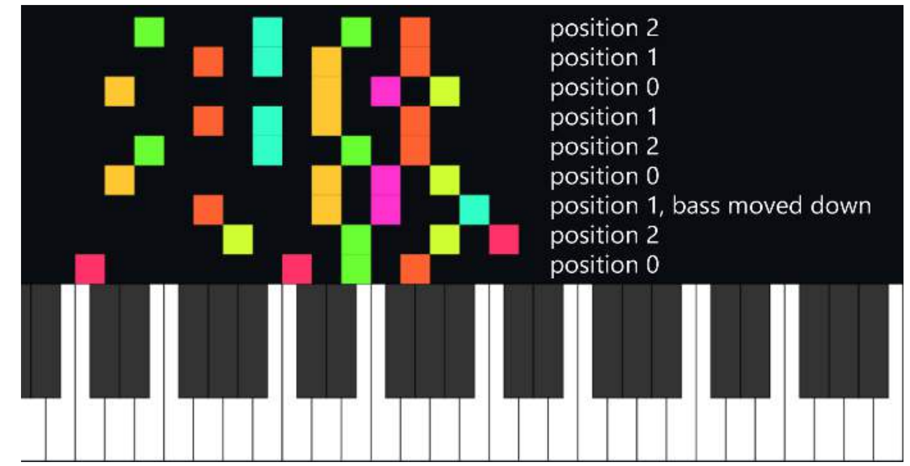
This algorithm's pretty compact! The core JavaScript code from this article's website for this fits in this sidebar:

```
If (nextPosition == 0)
  chord = [n-11,n-4, n-2, n];
else if (nextPosition == 1)
  chord = [n-7, n-5, n-3, n];
else
  chord = [n-9, n-5, n-2, n];

if((chord[0] % 7) == 6)
  chord[0] -= 2;

nextPosition =
  Math.floor(
    nextPosition + 1
    + Math.random() * 2
  ) % 3;
```


Here's an example of a harmonization generated using this algorithm on the melody above.



Here we've chosen each new position randomly, but we could choose it deterministically instead. We have a choice of 3 positions for the first note and 2 positions for each subsequent note, for a total of $3\ 2^{m-1}$ possible harmonizations of an m-note melody. Knuth also shows how to steganographically encode information this way, as a stream of a base-3 integer followed by m-1 bits.

This algorithm comes from Chapter 22, *Randomness in Music*, of Donald Knuth's book *Selected Papers on Fun & Games*, where he attributes it to a 1969 class from David Kraehenbuehl (1923-1997) at Westminster Choir College. I've rephrased it a bit in the presentation above.

Procedures for creating harmonies have existed for a while, although they're usually phrased as a set of constraints. The extra step of choosing a random harmony that satisfies the constraints sometimes turns it into an algorithm in the usual sense.

A set of constraints can also be a program! One can write a Sudoku solver in Prolog by specifying the rules of Sudoku and the initial clues as CLP(FD) constraints; Prolog's constraint solver will find a solution.

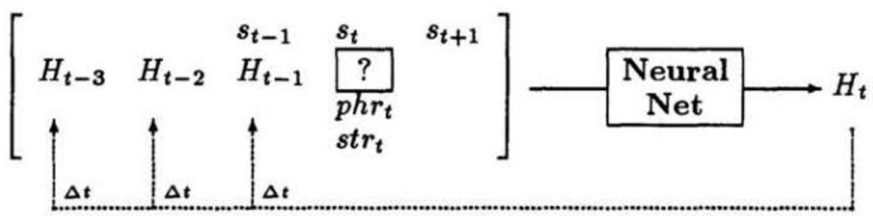
For instance, Johann Joseph Fux's 1725 *Gradus ad Parnassum* describes a set of rules for constructing each of four kinds of counterpoint, such as beginning and ending on consonance, avoiding tritones, and avoiding parallel fifths and octaves.

It turns out that the first few chapters of Peter Ilyitch Tchaikovsky's *Guide to the Practical Study of Harmony* have guidelines that come relatively close to describing Kraehenbuehl's algorithm above! Section 9 of (from the 1900 English translation) describes the rules above for the three triad positions above and the bass part (although it doesn't include the rule for moving the bass note to create a dominant 7th chord). §14 describes how Kraehenbuehl's algorithm never uses the same position twice in a row in terms of avoiding parallel fifths and octaves, with the constraint that

Two triads, however closely related internally and externally, must never directly follow each other in the same position, as parallel fifths and octaves must necessarily occur.

In addition to algorithms like the ones described above, some musical harmonization algorithms take a corpus of existing harmonized pieces, then try to construct a statistical model of what harmonies in the corpus tend to look like. They can then generate harmonies for new melodies by choosing from the distribution of harmonies they think are likely.

Hild, Feulner and Menzel (1991) trained a system named HARMONET on a collection of 400 Bach chorales. The neural network (or neural networks – it can use one, or choose between the outputs of three) tries to guess the next symbolic harmony given the previous harmonies, the previous, current, and next notes, and the rhythmic location of the note in each bar.



From HARMONET: A Neural Net for Harmonizing Chorales in the Style of J. S. Bach

More recently, David Li's *Choir* and *Blob Opera* are web applications that also generate four-part SATB harmonies using a neural network. There doesn't appear to be much information about the neural network they use (or, worryingly, about the dataset it was trained on!), but its input and output format looks similar to Liang, Gotham, Johnson, and Shotton's BachBot (from *Automatic Stylistic Composition of Bach Chorales from Deep LSTM, 2017)*.

Both Choir and Blob Opera use the same neural network – or, at least, the same weights. Blob Opera has an additional network to synthesize sound.

BachBot uses a Long Short-Term Memory (LSTM) architecture, which essentially is a deep neural network that transforms vectors to other vectors while keeping some internal memory state.

BachBot's source code is available online at https://github.com/feynmanliang/bachbot.

BachBot takes as input a series of 16th-note *frames*, each of which contains a melody note (continued or not from the last frame) or a fermata (denoting the end of a musical phrase). These are embedded into vectors and passed one by one to the LSTM, which outputs a probability distribution. Liang et al. then optimized the parameters of the model so that it tended to assign high probabilities to the full harmony of the Bach chorale for that frame. When running on new melodies, BachBot inputs frame and melody symbols to the LSTM, retrieves the note probability distribution output by the LSTM, and chooses the harmony consisting of the highest-probability notes. If we wanted BachBot to make more unexpected decisions, we could instead have it randomly sample from the probability distribution.

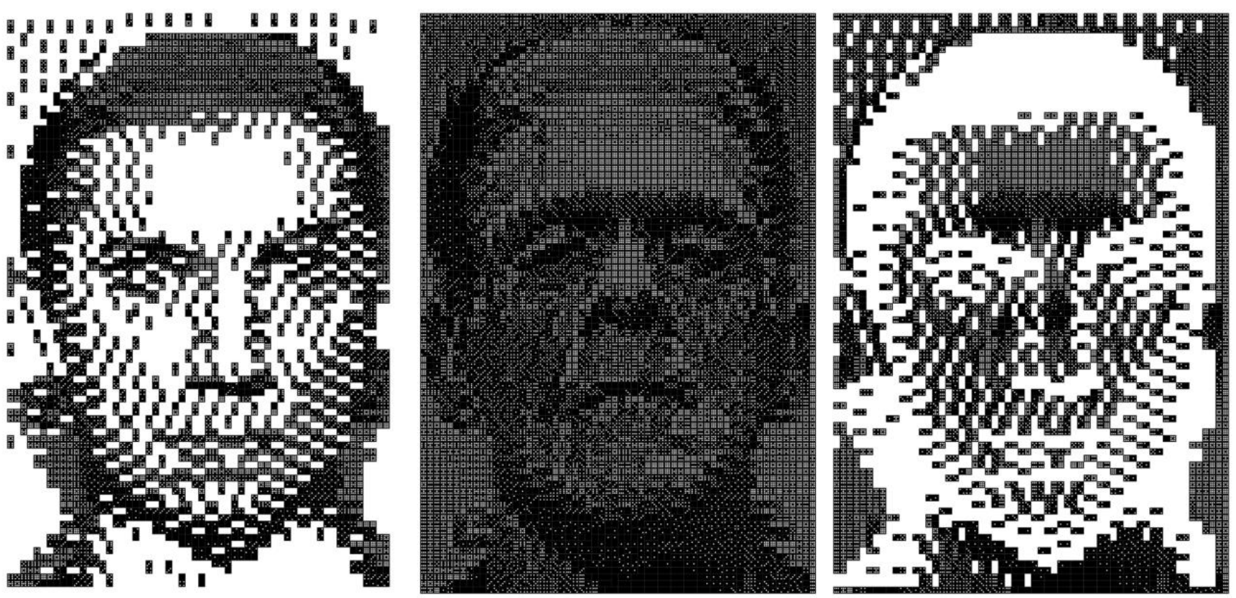
However, It's possible to miss in the above discussion that constructing harmonies is an art. There are many harmonies Kraehenbuehl's algorithm above can't construct. The neural network-based methods would be unlikely to guess surprising but sublime melodies, and also usually have no way to artistically collaborate with a user. The algorithms here also generally have no concept of the emotions of the piece – Kraehenbuehl's algorithm will choose a random harmonization at each step!

Breaking melodic patterns can be a powerful tool. Additionally, books on harmony sometimes have conflicting guidelines, and while that defies computer implementation, that's okay: Tchaikovsky, for instance, writes of weighing which conflicting rules to follow, or of breaking earlier rules artistically.

Domino Steganography: Dracula + (-Dracula) = Frankenstein's Monster

Robert Bosch

A triptych of domino mosaics. The center piece arranges 48 complete sets of double-nine dominos in such a way that they collectively resemble Boris Karloff as Frankenstein's Monster. The left piece reveals what happens when we remove all of the horizontally oriented dominos; the remaining dominos collectively resemble Bela Lugosi as Count Dracula. The right piece shows what happens when we remove all of the vertically oriented dominos; here, the remaining dominos collectively resemble the negative of the Bela Lugosi image. Hence the title: Dracula + (-Dracula) = Frankenstein's Monster.



Dracula + (-Dracula) = Frankenstein's Monster Robert Bosch G4G14

Sakura Pythagorean Tree

Marcela Chiorescu

Inspired by the Pythagorean Tree in a plane, I stitched the Sakura Pythagorean Tree on a temari ball of diameter 58 cm. This Pythagorean tree has order five and begins with a square of side 5 cm. Upon the first square I constructed two squares to depict the Pythagorean theorem, and from there I continued recursively. Each square is scaled down by a linear factor of about 0.7.

A G4G14 Heartwork

Matjuska Teja Krasek

An artwork dedicated to the memory of Marc G. Pelletier (1958 – 2017), Robert Abbott (1933 – 2018), and Markus Götz (1974 – 2018).

Ampedhead – Microdose: Peach, Peach, Peach

Matjuska Teja Krasek & Stephen Russell

GATHERING FOR GARDNER 14 EXCHANGE GIFT

Authors: Teja Krašek (tejak@yahoo.com) 1

Stephen Russell (bdoc23@gmail.com) 2

Organization1: Freelance Artist

Organization2: Barefoot Doctor World

Title: AMPEDHEAD - Microdose: Peach, Peach, Peach

Our **AMPEDHEAD - Microdose: Peach, Peach, Peach** audio-visual G4G14 exchange gift features a kaleidoscopic animation by Teja Krašek intervowen with the unique multi-layered, cinematic electro-organic AMPEDHEAD composition. The soundscape comprises advanced sound-frequency design (68 Hz. 110 Hz, 300 Hz, 432 Hz, and 528 Hz) to produce binaural beats and activate healing. With an inaudible subliminal affirmation pattern and unique combination of elements involved, it makes it the only brain entrainment technology of this kind on the planet. Our video can be watched at the following link: https://www.youtube.com/watch?v=omr06epqm0I

G4G14 Gift
The Meta-Hilbert Curve → 14 Hilbert Curves

Doug McKenna

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Boulder, Colorado USA

doug@mathemaesthetics.com

Abstract

The Meta-Hilbert Curve construction is an infinite sequence C_n of space-filling curves, each a continuous mapping from the unit interval onto every point within a rotationally symmetric fractal tile, each the same area. The boundary of each C_n comprises eight "fractalized" Hilbert Curve approximation paths, four facing outwards, four inwards. The sequence of boundary fractal dimensions converges to 2.0, i.e., space-filling. C_∞ has twice the area of any C_n, but comprises 14 piecewise-connected Hilbert Curves, some coinciding with others.

The number 14 is not normally associated with any mathematical object demonstrating four-fold rotational symmetry in the plane. But due to combinatorial constraints arising from the need for open-ended, piecewise-connected continuity, and the interesting things that can sometimes happen when one takes a limit, the Meta-Hilbert Curve comprises $6 + 6 + 2 = 14$ Hilbert Curves. The second 6 lie, in reverse order, on top of the first 6.

The following is a condensed version of a much fuller exposition and motivation, with animations of the construction and interactive drawings, taken from my dynamic, electronic book, *Hilbert Curves* [1], currently published for iPad and iOS devices, and M1 MacOS computers, demoed at G4G14. The construction was first minimally described (under the name "Inside-Out Curve") in a paper on loops in DNA [2]. My collaboration with primary authors of that paper began with a chance meeting at the G4G10 Gathering for Gardner conference.

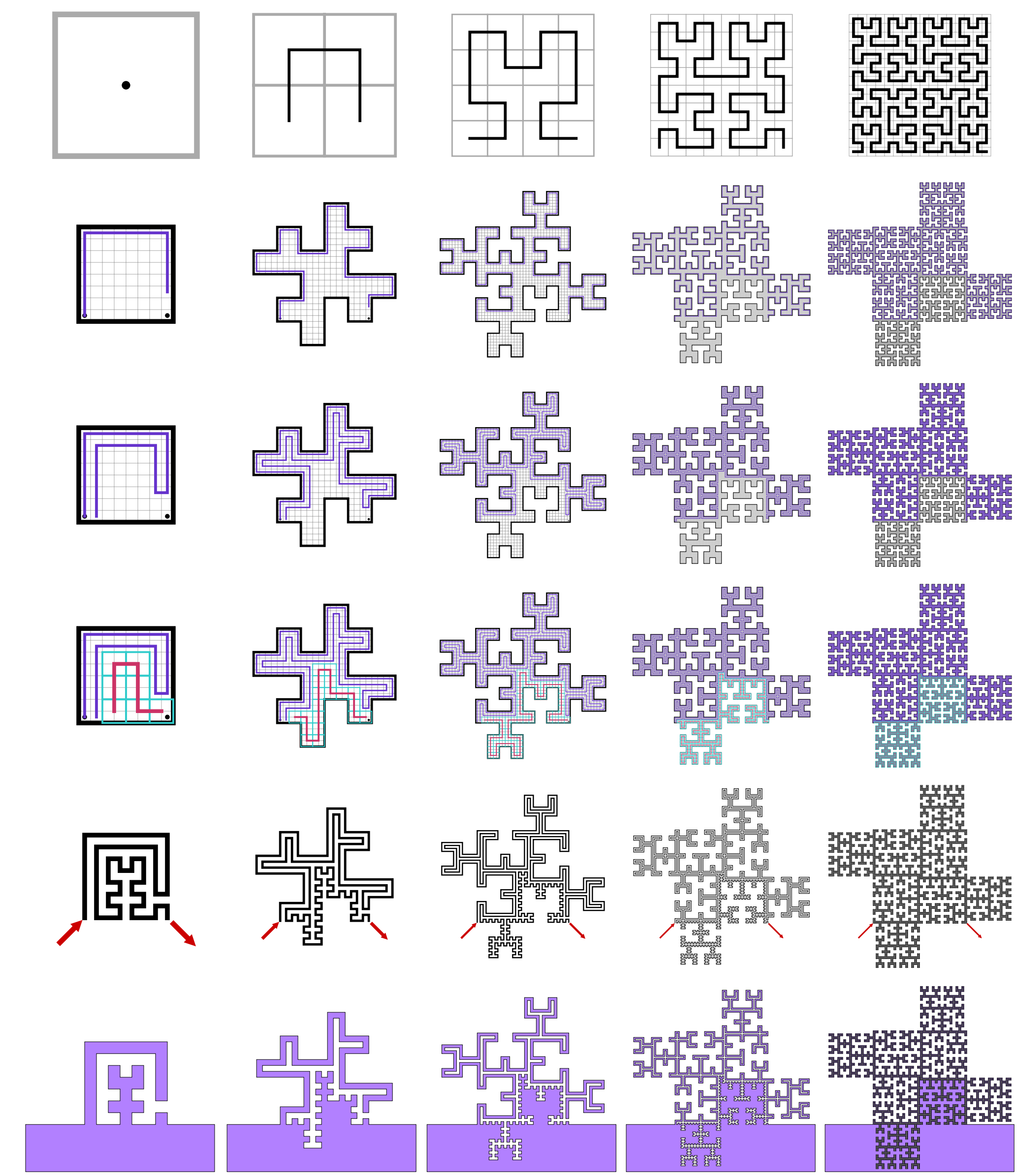


Figure 1: The Meta-Hilbert Curve Construction

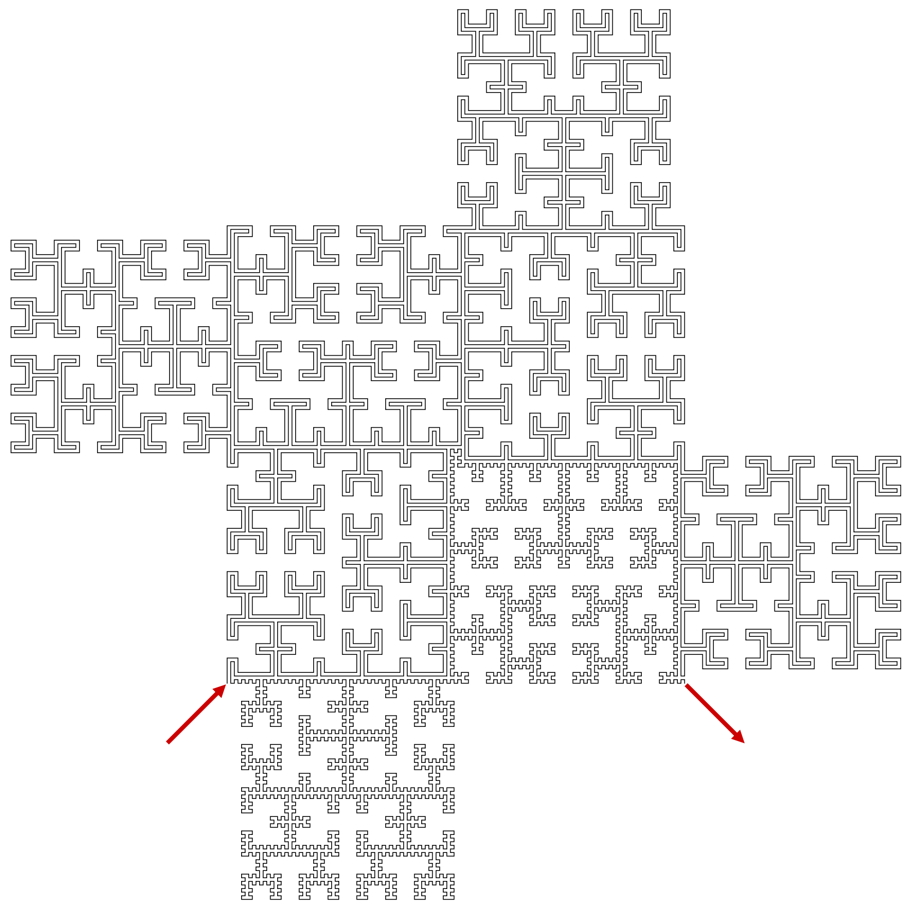


Figure 2: The Meta-Hilbert Curve motif for C_4 threads its prototile from a (left arrow) to b (right arrow) in such a way that it hews within ϵ (one sub-square width) of the outside or the inside of six Hilbert Curve approximation paths, two up (facing right, then left), two over (facing down, then up), and two down (facing left, then right). Within 2ϵ of b, the path turns around and hews within 2ϵ of the same six approximation paths, but in reverse order. Then, from within 2ϵ of a, the path completes its journey to b by hewing within 2ϵ along the bottom two Hilbert Curve approximation paths (facing down, then up), vibrating as a square-wave would. Scaled, rotated, and connected copies of this path would build the second approximation of this motif's self-similar, space-filling curve. The eventual fractal tile's boundary's dimension increases to 2.0 (with $\epsilon \to 0$) as increasingly detailed Hilbert Curve approximation paths are relied upon to build each fractal tile C_n. The square wave vibrations disappear in the limit, so the mapping to the final two Hilbert Curves has half the instantaneous "speed" of the previous 12 Hilbert Curves.

Approximation paths to the Hilbert Curve illustrate the sequence of always edge-adjacent sub-squares that increase in number as the square of the reciprocal of their decreasing-to-zero size. The sub-squares converge to the continuous Hilbert Curve, a fractal of dimension 2.0.

But the same is true of the "fractalized" boundaries of C_n. The connected area of each C_n becomes increasingly wispy, elongated, and branched, while the tile boundaries become increasingly close to space-filling Hilbert Curves. The interior area vanishes at the same limiting "moment" that the points to which the boundary is converging from both inside and outside take over being the area of a rotationally symmetric tile (with linear boundary). So if $A(n)$ is the normalized unit area of every C_n, we have $A(n) = 1$ for all finite n, but $\lim_{n\to\infty} A(n) = 2$.

The foregoing Meta-Hilbert construction works using any order-n approximation path for any square-filling generalization of the Hilbert Curve based on the $n \times n$ recursive subdivision of the square; see, e.g., Figure 102 in [1], which uses an order-3 Wunderlich Curve approximation path to build an element of a Meta-Wunderlich Curve sequence of space-filling curves.

References

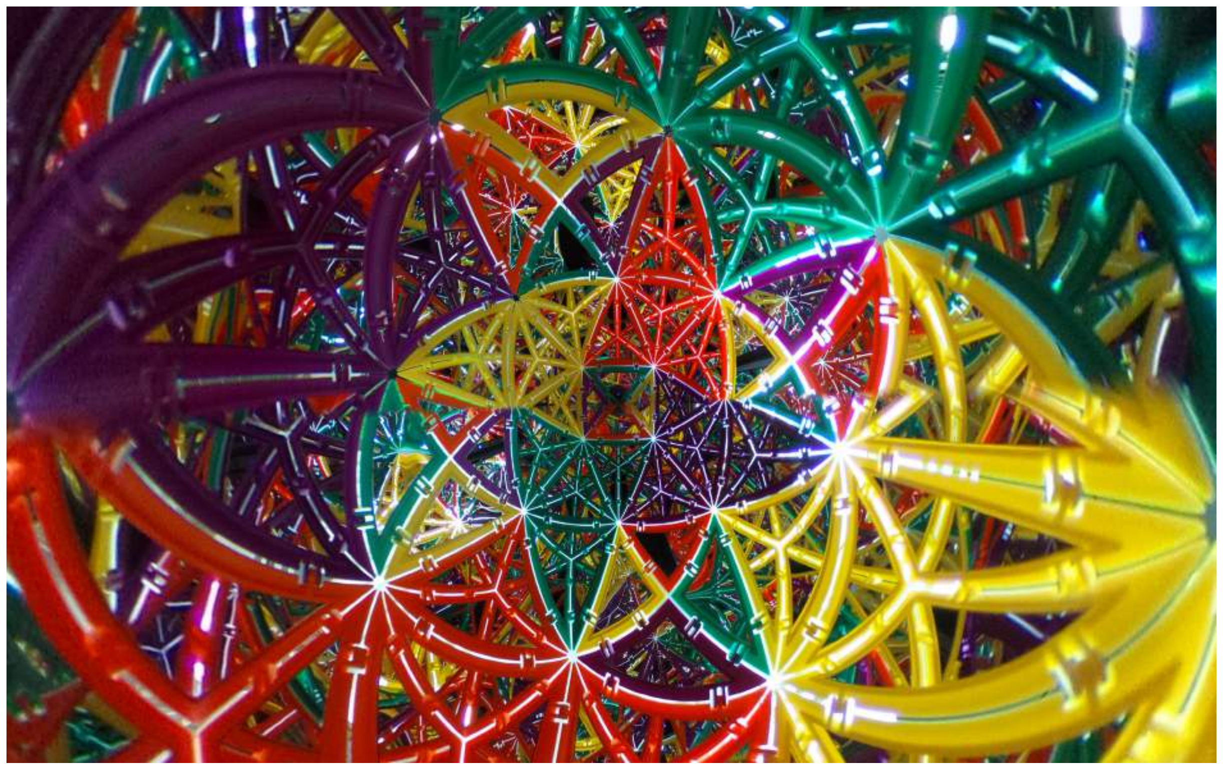
[1] McKenna, D. M., *Hilbert Curves: Outside-In and Inside-Gone*, Mathemaesthetics, Inc. (2019), ch. 6. ISBN: 978-1-7332188-0-1 (iPad/iPhone/MacOS eBook available at https://apps.apple.com/us/app/hilbert-curves/id1453611170 (as of 4/14/2022).

[2] Sanborn, A. L., Rao, S. S. P. , et al, "Chromatin extrusion explains key features of loop and domain formation in wild-type and engineered genomes," Proceedings of the National Academy of Sciences, **112**, p. 47 and Fig. S3 (Nov. 24, 2015). See https://www.pnas.org/doi/10.1073/pnas.1518552112 (as of 4/14/2022).

[3] Sloane, Neil A. J., *The On-Line Encyclopedia of Integer Sequences*, accessible electronically at http://www.oeis.org/search?q=A000532 (as of 4/14/2022).

Snub cube kaleidoscope

Bjoern Muetzel and Matthew Bajor
Eckerd College, St. Petersburg, Florida



Description:
The snub cube is one of the Archimedean solids. We made a stellated version of this solid and put mirrors on all of its inside faces. We then used a spherical camera to take a picture of the inside and transformed it using a stereographic projection. This method produces characteristic kaleidoscopic images for each Platonic and Archimedean solid. The snub cube and the snub dodecahedron are the two chiral solids. Special for these two solids are the spirals in the image.

More info: http://natsci.eckerd.edu/~muetzeb@campus/gallery.php

Polygoriginal

Maria Samuelson & Moses Samuelson-Lynn

Polygoriginal (Poly-gor-ig-in-al) is a program that generates a personalized geometric art image with an algorithm I am writing. It varies, for example, the number of vertices, the hex colors used, and the spacing of the vertices based on unique inputs such as your name, your favorite integer (between -65537 and 65537 inclusive), the floor of the number of years old you are (colloquially called your age), and the number of G4GNs that you have attended. It is has a small easter egg that if you input “Martin Gardner” as your name (it doesn’t matter what the other inputs are), it will give you the G4G14 logo.

Included below is a link to two sample images (aesthetics may be subject to change).

https://www.dropbox.com/s/ho3h9f4gh070h68/Polygoriginal%20Examples.png?dl=0

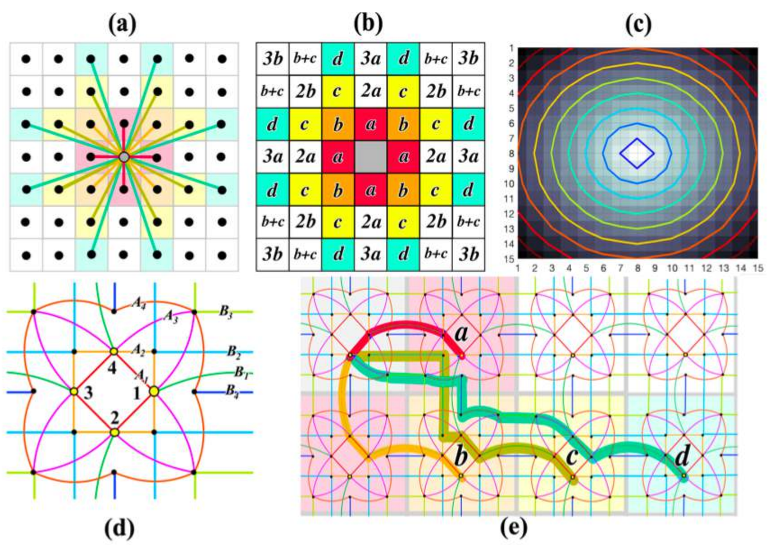
Designed and executed by Moses Samuelson-Lynn with the supervision and advice of Maria Samuelson

Waving Goodbye (Artwork) - by Art Terlep

This gift of 4 4x6in images will be included in the physical exchange: Martin Gardner, Elwyn Berlekamp, John Conway, and Richard Guy. This digital version also features Les Shader.

If you've ever traveled in a grid, you might have noticed that the areas you can reach for a fixed time form a diamond shape. For example, if you can travel 6 blocks, you might go north 6 and 0 east, or 3 east and 3 north. These lines of equal distance, or iso-distance curves, annoy my sense of Euclidean geometry and personally, I like to have more circular distances whenever I play a board game on a grid. Well, chamfering is one solution in which weighted edges are drawn to nearby neighbors to approximate Euclidean distances. In the image (a) below, red = 1, orange = sqrt(2), yellow = sqrt(5), and so on. (c) Shows the iso-distance curves. So, you can generate pretty roundish minimum distances this way!

Terlep-Art-GiftExchange-WavingGoodbye-G4G14

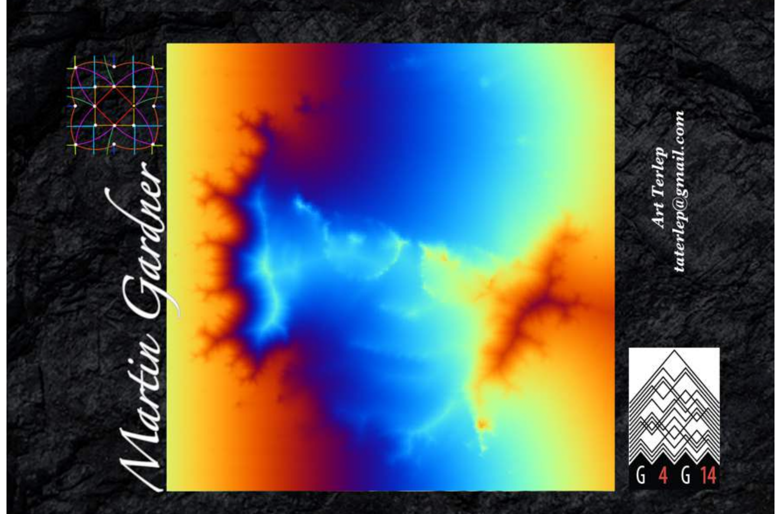


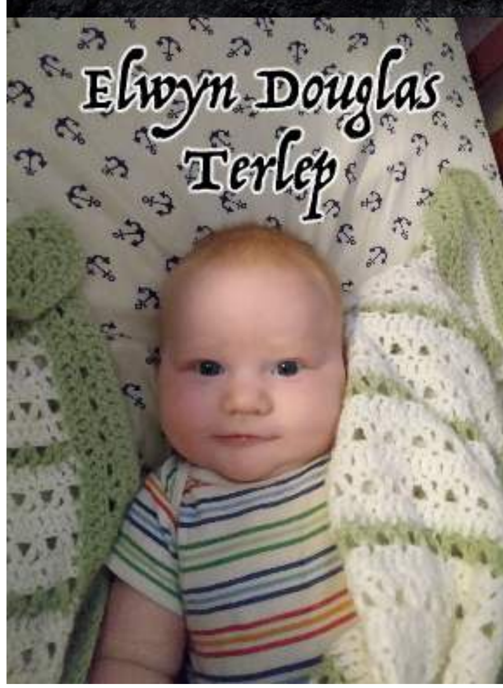
Well, I found another way to make the same chamfer with a special kind of graph (network) called a replacement product. Basically, instead of drawing lines out from a single point, a little network is used inside each pixel to "process" the Euclidean distance through a handful of channels (some of you may be noticing at this point that there's an eerie similarity to my PUZZLE gift here and you aren't wrong!). The catch is that you start and stop on the same vertex (node). The graph in (d) has 4 operating modes, or options for the start/stop point (yellow dots), which when composed together give a wavy, shadowed version of the original. Although it's constructed similarly to a kind of Fourier transform, it actually sort of puts you *back* in the image space (isn't that spooky?). By using the phases to select color gamuts and adjusting the frequency over the image space, surreal artistic interpretations of the original image can be imagined. I decided to apply this to the images of some familiar faces and that is my gift to you.

One other thing, I owe my entire G4G experience to Leslie Elwyn Shader and his family and this work is inspired by conversations with Elwyn Berlekamp over 10 years ago regarding the concept of "influence" in the game of Go which occurred some time after he gifted me an entire Go board - a great story you should ask me about sometime. For them, I have a very special gift, and that is the name of my fourth child, Elwyn Douglas Terlep (though I admittedly found out after the fact that Elwyn was also Les's middle name! The infinitely improbable odds!). Questions about his rather unique name always give me a reason to talk about some great people in my life and the mathemagical conference that inspired so many things over my years of absence from it.

If you want to know more about my clown car chamfers, just email me and I'll send you copies of my recent work in this area! I think it's a lot of fun and I hope you do too. It's good to be back. I missed you all.

Please contact me at taterlep@gmail.com for additional (free) print copies of the original or the raw image or .mat file. Cheers!





Modern Interpretations of Traditional Islamic Geometric Patterns

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My Background

I have been creating geometric designs and models since junior high school. During a trip to India in 2012 I became especially enthralled with Islamic geometric patterns. I began pursuing my art more fully, and I am now working full time as an artist. I have been an active participant in the Bridges conference since 2013, and am excited to attend my first G4G conference this year. All of the works shown here, and many more, can be found on my website.

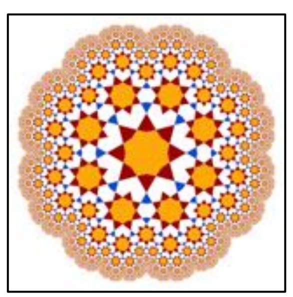
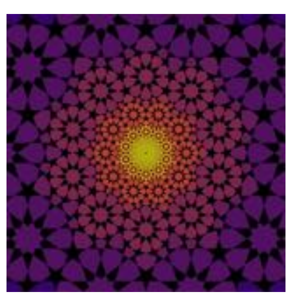
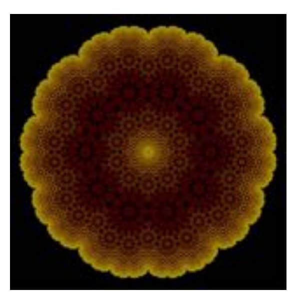
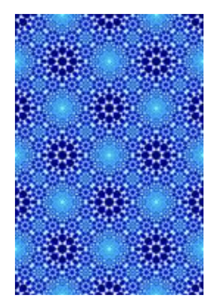
My Recent Work

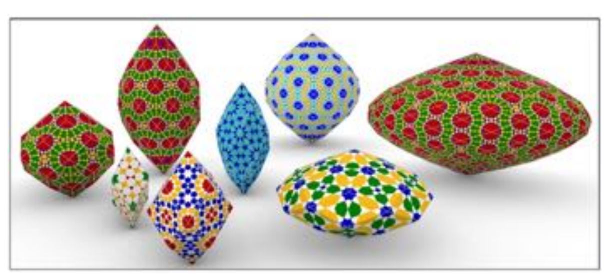
In the past several years I have been exploring ways of combining Islamic geometric patterns (hereafter, IGP) [1] with various mathematical concepts to create meditative, contemporary art and décor.

I will briefly present three avenues of exploration here:

1. Arranging IGP motifs in fractal arrangements
2. Wrapping IGP around Platonic Solids
3. Applying IGP to the faces of polar zonohedra

A Sneak Peek at What's Ahead…

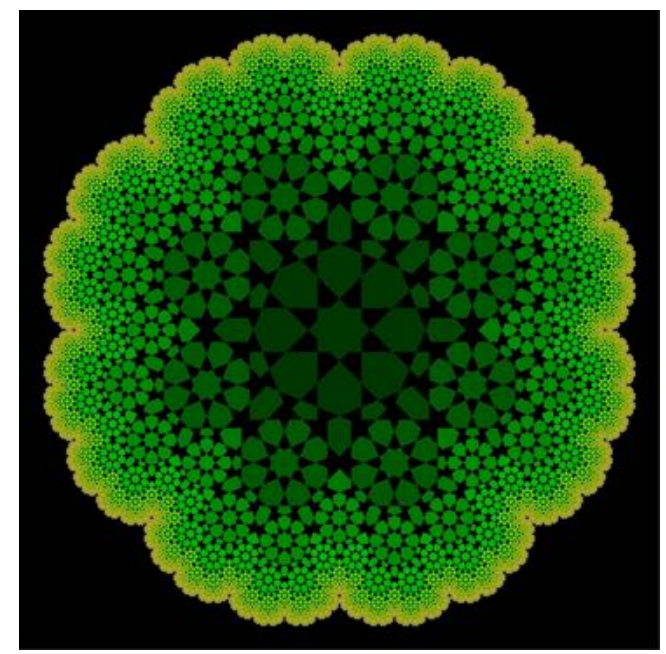





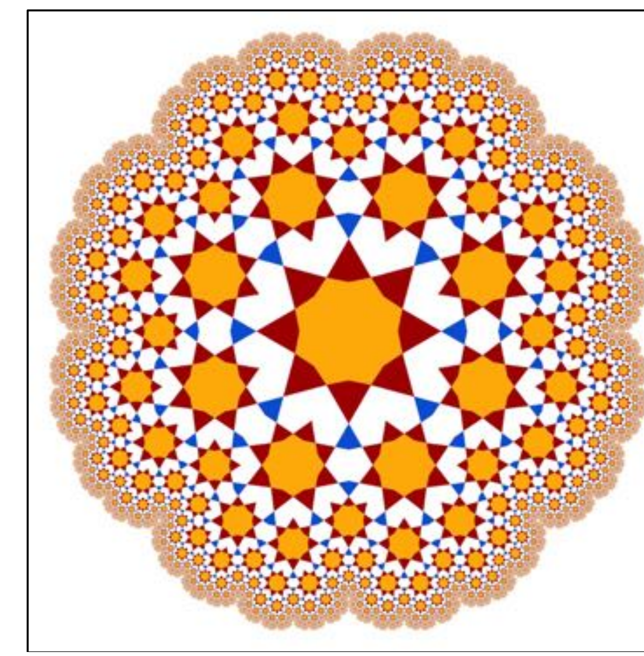



IGP + Fractal Trees

My first exploration upon returning from India was to find a way to arrange IGP in a fractal pattern, i.e., with motifs at infinitely many scales and with self-similarity throughout the pattern. Ultimately I devised a way to arrange motifs at the nodes of n-fold fractal trees, connect the motifs from level to level in a way consistent with traditional patterns, and then arrange such trees radially (with selective pruning) into radial designs (see my Bridges 2013 paper for details [2]).

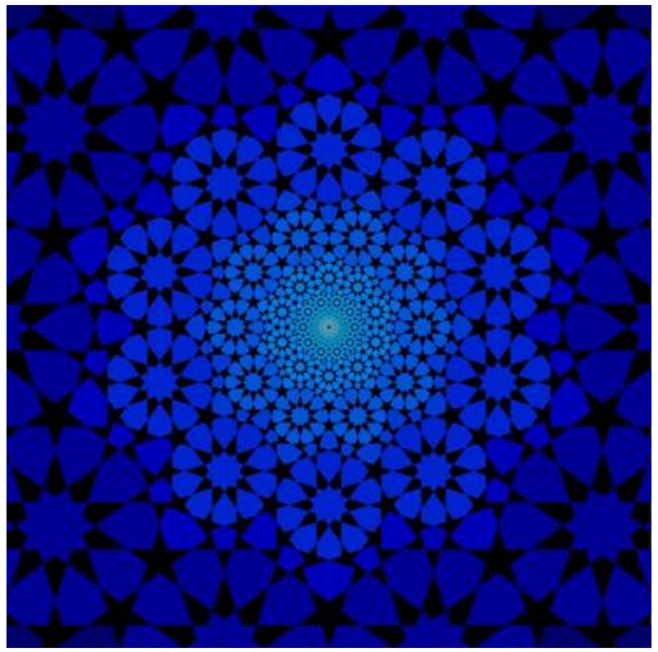
Outward Radial Patterns

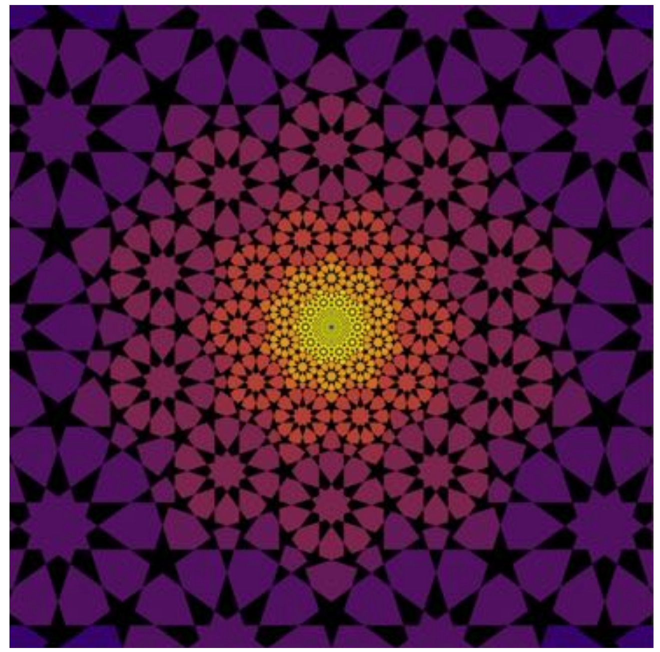

Infinity Bloom 8 - Forest Green/Yellow


Infinity Bloom 8 - Moroccan on White

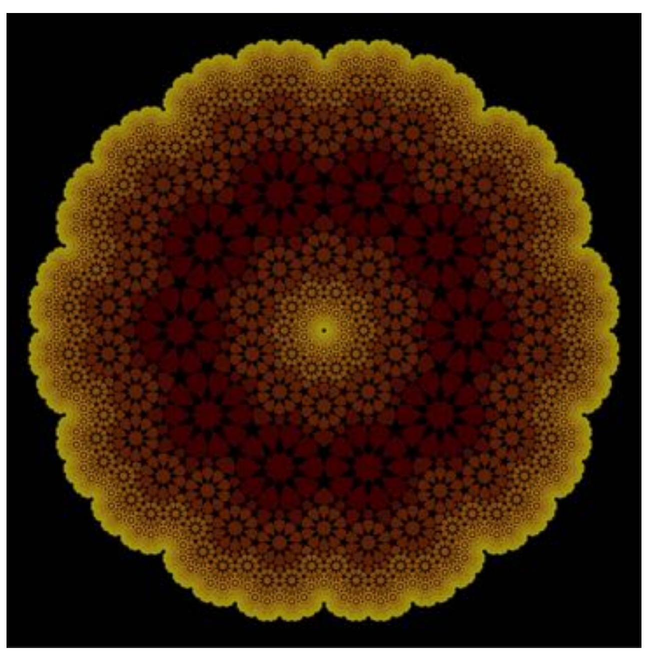
Points of convergence on the peripheries of these designs suggested patterns that shrank radially inward, instead of outward, and the two can be elegantly combined into an "inward-outward" pattern as well.

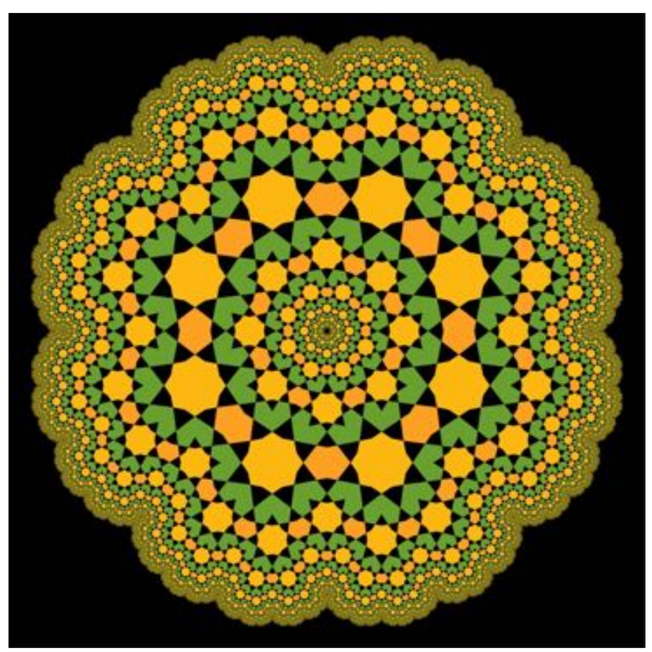
Inward Radial Patterns


Starburst 10 - Blue


Starburst 10 – Plum/Honey

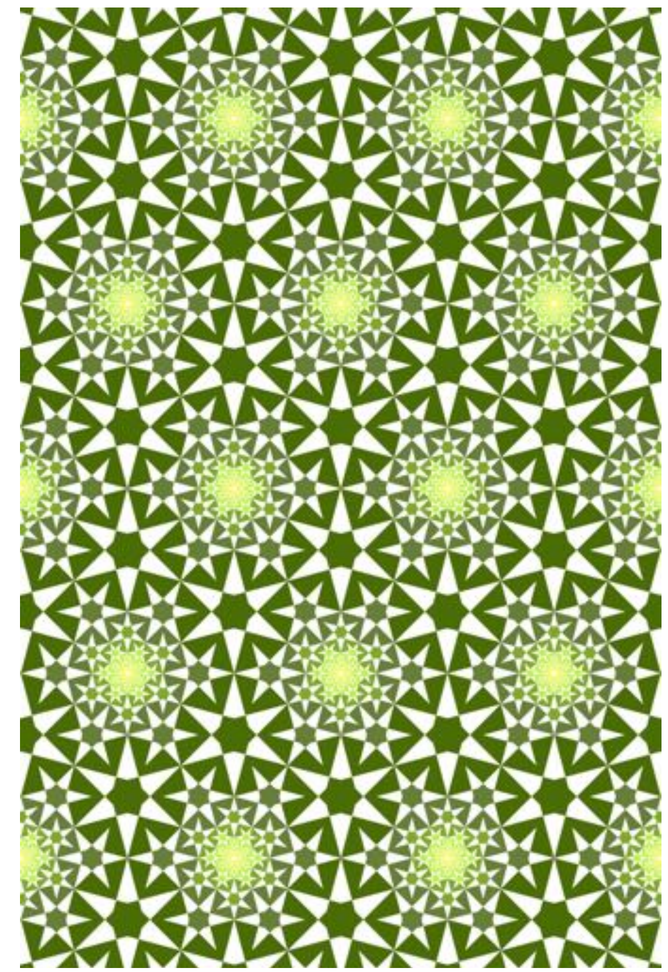
Combined ("inward-outward") Radial Patterns

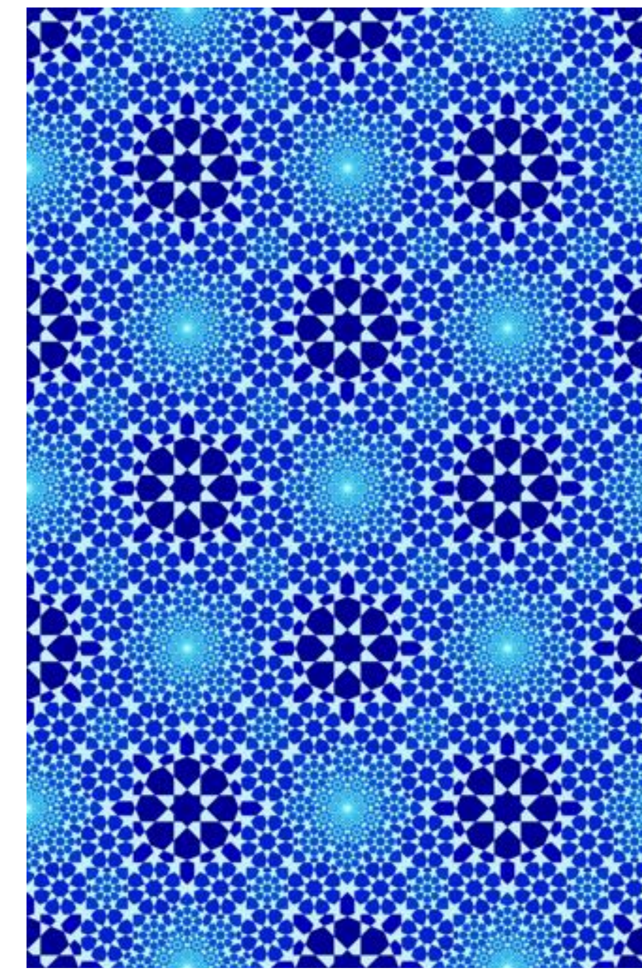

Infinity Bloom 10 – Garnet/Honey


Infinity Bloom 8 – Emerald/Honey

Finally, some such patterns lend themselves to periodic repetition in the plane, coming full circle back to the original aesthetic but with "embedded" areas of fractal diminution.

Repeating Patterns


Vibration 6 – Moss/Yellow


Vibration 8 - Blue

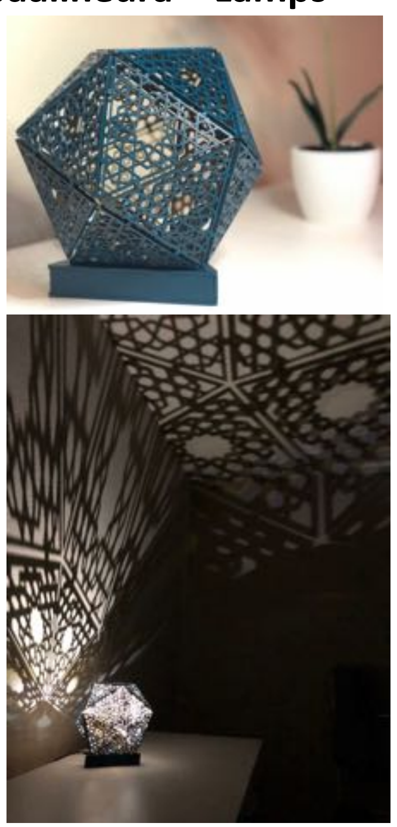
IGP + Platonic Solids

Lamps

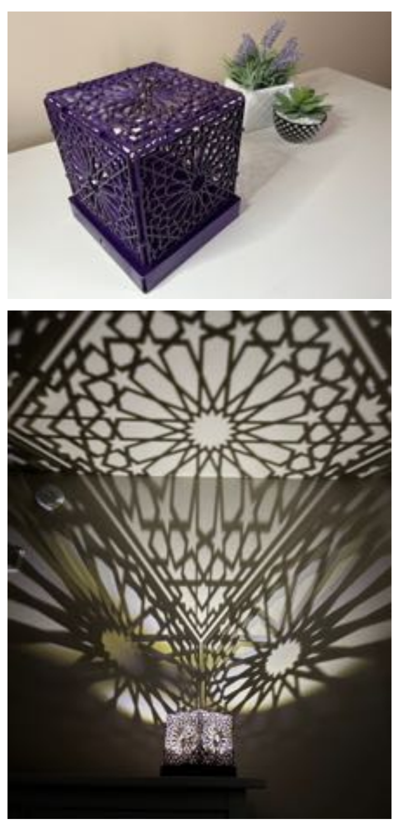
Since many IGP repeat on grids with local repeat areas of equilateral triangles, squares, and pentagons, it is an easy conceptual leap to take these repeat areas and apply them to the surface of the Platonic solids. Many others have followed this approach to create solids with beautiful surface patterns.

What I have done which is slightly different than most is to focus on cutting the patterns through the faces. This idea was directly inspired by the carved sandstone screens featured prominently in the forts and palaces of Northwest India, known in Hindi as *jaali* (meaning "net") [3]. Thus I call my shapes that combine this idea with polyhedral shapes, Jaalihedra™. My most recent series of work has been a number of Jaalihedra™ Lamps, of which a few are pictured below. As you can see, they function as sculptures in their own right when unlit, but cast dramatic shadows when lit.

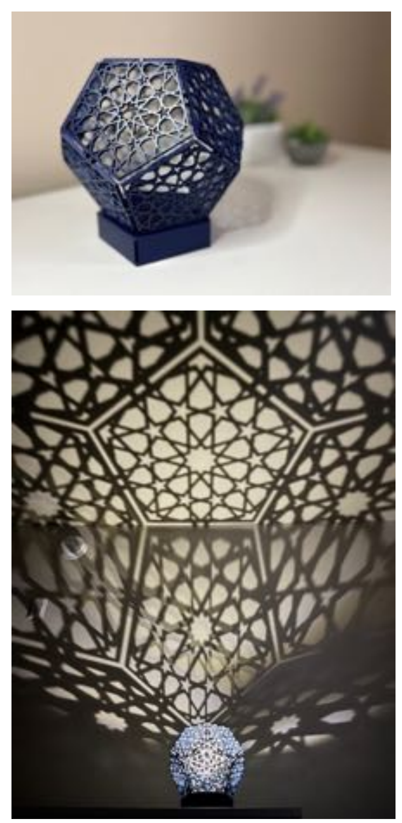
Jaalihedra™ Lamps



"Hex"



"Mexuar I"



"Pentastar I"

Sculpture

A different approach is to custom design face patterns that would not necessarily tile the plane in the original repeat pattern. As long as the edges and vertices of each face are compatible with adjoining faces, the resulting patterned solid can still be cohesive and attractive.

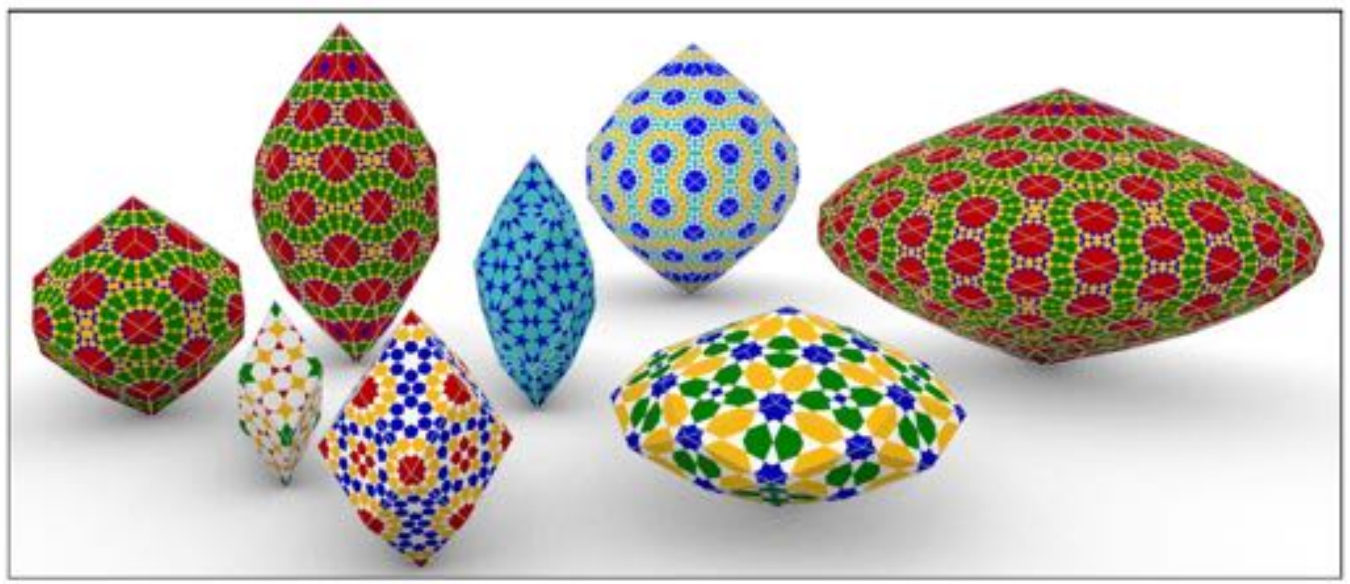
In the middle of 2020 – in order to celebrate in the advance the end of that oh-so-trying year! – I decided to embed "2021" into a piece of art. Ultimately, I designed a custom IGP that placed 7 petals of a rosette at each corner of a pentagon, thus yielding 21-pointed rosettes at each of a dodecahedron's 20 vertices where 3 faces meet — hence, 20(21). Furthermore, placing 20-fold rosettes in the center of each of the 12 pentagonal faces yielded (20)12 – the year I first started investigating IGP in depth. The resulting sculpture was meticulously built using hand-painted wood and laser cut mat board, and is shown below (as well as detail of one face on the cover page).

(20)12-20(21): Ten Years of Inspiration

IGP + Polar Zonohedra

In addition to the repeat areas listed above, rhombic repeat areas are also quite common in IGP. There is a class of beautiful polyhedra called polar zonohedra (hereafter, PZ), all of whose faces are rhombi of various aspect ratios [4].

I have a forthcoming Bridges paper [5] in which I discuss how to identify PZ whose face angles are "close enough" that IGP with N-fold local symmetry can be applied to all of the various faces of a single PZ in a cohesive manner with minimal distortion. The examples shown below are digital models, but I plan to execute many of these as physical sculptures and lamps in the coming months.



A variety of polar zonohedra decorated with rhombic-repeat IGP

Conclusion

The tradition of IGP is so vast and rich that there are countless opportunities for modern artists to expand these patterns in new directions. I will continue to explore these horizons, and encourage anyone who finds these patterns as captivating as I do to embark on explorations of their own!

References

For those interested in more information on some of the topics mentioned here, see:

[1] Islamic geometric patterns - https://en.wikipedia.org/wiki/Islamic_geometric_patterns

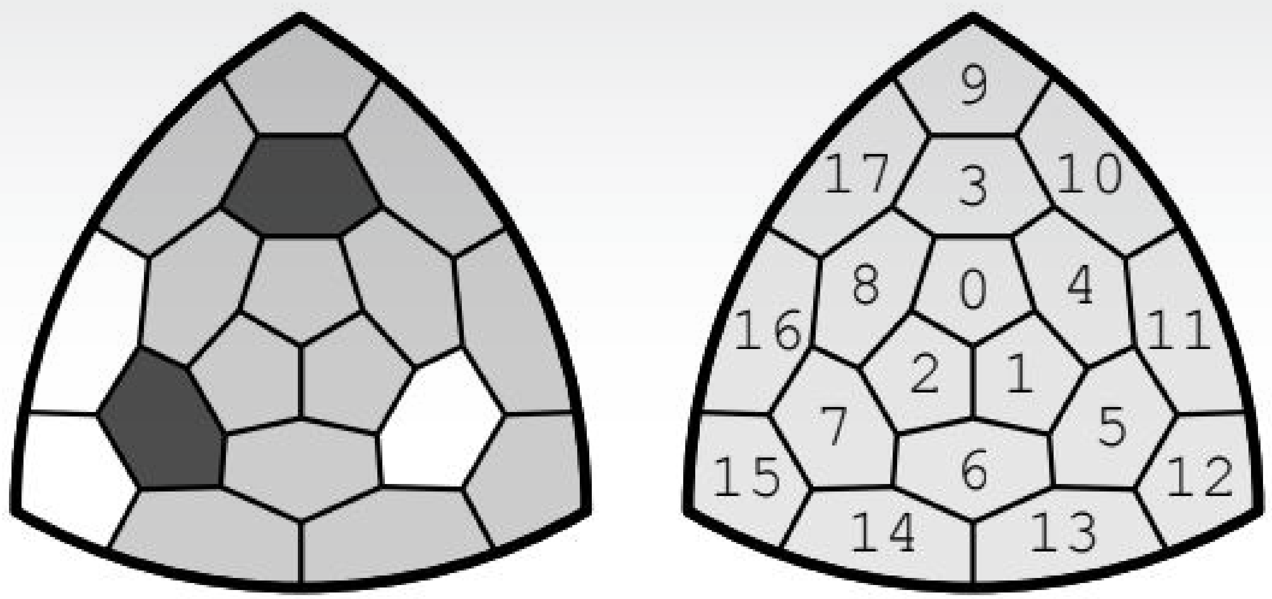
[2] Fractal Islamic patterns - http://archive.bridgesmathart.org/2013/bridges2013-87.html

[3] Jaali - https://en.wikipedia.org/wiki/Jali

[4] Polar zonohedra - https://archive.bridgesmathart.org/2021/bridges2021-7.html

[5] IGP on polar zonohedra - https://archive.bridgesmathart.org/ (exact URL TBD; look for year 2022 and/or search this page for "Webster"

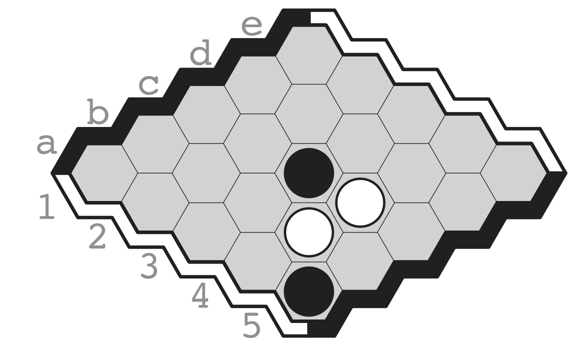
GAMES



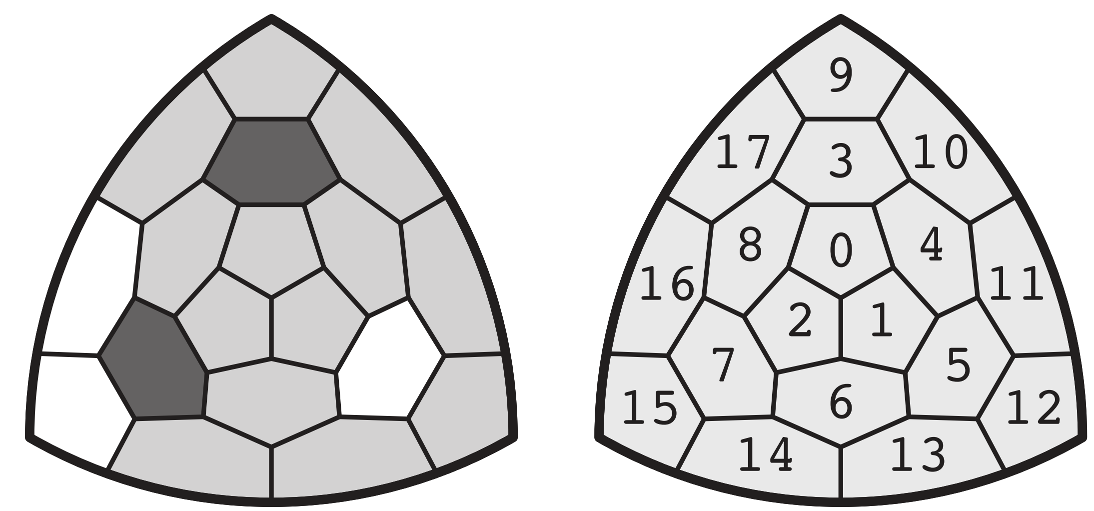
Hex: Two Books and One Puzzle | Ryan Hayward | Page 48

G4G gift exchange. Two puzzles, hex and Y.

submitted by Ryan Hayward email hayward@ualberta.ca



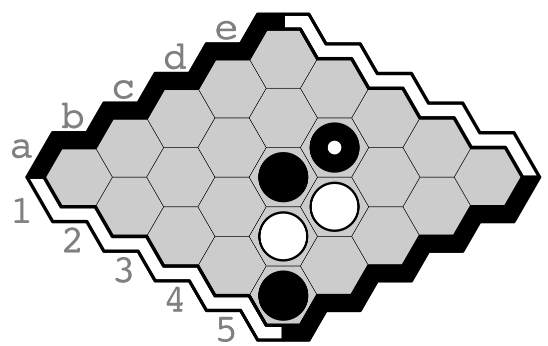
hex puzzle based on a 1942 puzzle by Karen Thorborg. Rules: on a turn, color any empty cell with your color; win by joining your two sides. **Find all winning next moves for B (black).** Hint: B's next move 5.B[c1] loses to 6.W[a4].

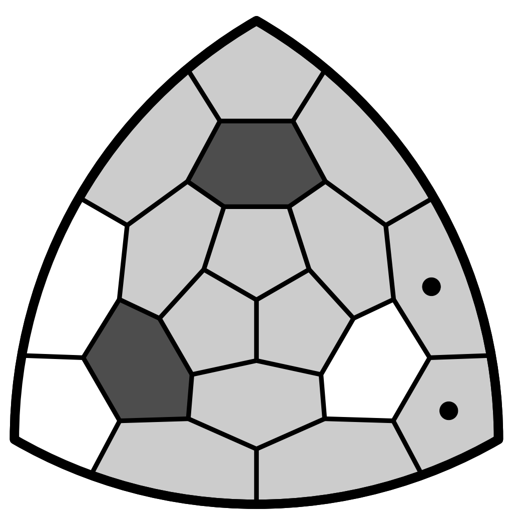


Y puzzle based on puzzles by Craige Schensted, later called Ea Ea. Rules: on a turn, color any empty cell with your color; win by joining all three sides. **Find all winning next moves for B.** Hint: each corner cell touches two sides, so 6.B[8] loses: 7.W[9] forces 8.B[17], then 9.W[10]! and W wins with one of {4,11} and one of {12, 13}.

Solutions on the next page (don't look).

More puzzles in *Hex, the Full Story*
https://www.routledge.com/hex-Inside-and-Out-The-Full-Story/Hayward-Toft/p/book/9780367144227
and *Hex, A Playful Intro* https://bookstore.ams.org/nml-54/ .





How To Get Even

Ryan Morrill

There are 27 counters in the centre pile. Players take turns claiming 1, 2, 3 or 4 counters from the centre pile. The player with an even number of counters when the centre pile is empty is declared the winner.

When we say we have *solved* a game, we mean we have found a **winning strategy**. What we mean by this, is we have found a strategy that wins every time, assuming both players play optimally. We introduce definitions for **winning positions** and **losing positions** in the following way.

- The empty game (or the game where you have lost and there are no more moves to be made) is a losing position.
- A winning position is one where **at least one** move sends to a losing position.
- A losing position is one where **every** move sends to a winning position.

Every position of the game is either a winning one or a losing one. If you can identify this, you will have the winning strategy from any position of the game. More discussion on winning strategies can be found in Berlekamp, Conway and Guy's wonderful series of books [1]. Let's try working backwards using this idea.

Solution:
The first key is to write out every position the game can be in. This includes the parity of each players hand. Here "even even" means both players have already taken an even number of counters, and "even odd" means the first player (one going next) has taken an even number of counters, and the other player has taken an odd number of counters. We write all viable positions as follows, up to 16 counters in the middle:

even even		1		3		5		7		9		11		13		15	
even odd	0		2		4		6		8		10		12		14		16
odd even	0		2		4		6		8		10		12		14		16
odd odd		1		3		5		7		9		11		13		15	

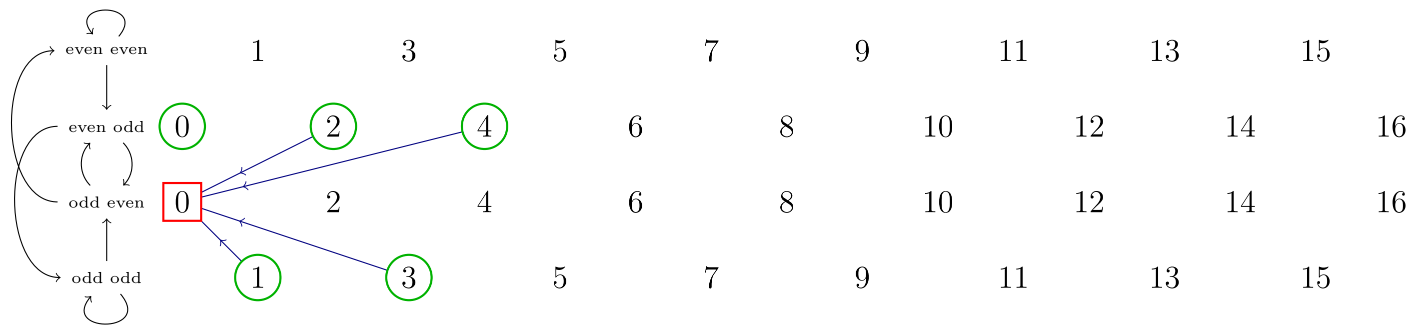
It is easy to see that if there are 0 left and you have even, then you have won, so this is a winning position. If there are 0 left and you have odd, this is a losing position. We put those down:

even even		1		3		5		7		9		11		13		15	
even odd	0		2		4		6		8		10		12		14		16
odd even	0		2		4		6		8		10		12		14		16
odd odd		1		3		5		7		9		11		13		15	

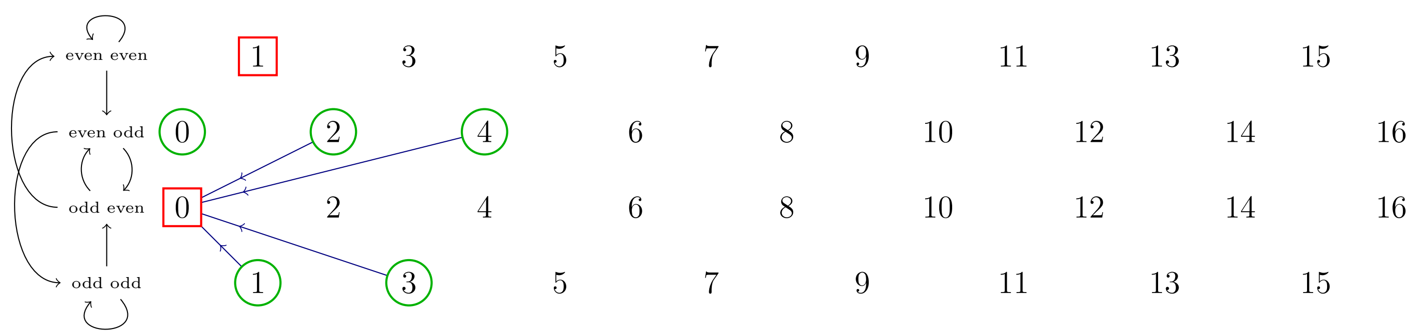
One important detail to be careful of: the board is from the perspective of the player going next, that means after each move, the "even odd" will swap. This means,

Counters taken	Even Number	Odd Number
In the First Row	Stay in the First Row	Go to the Second Row
In the Second Row	Go to the Third Row	Go to the Fourth Row
In the Third Row	Go the the Second Row	Go to the First Row
In the Fourth Row	Stay in the Fourth Row	Go to the Third Row

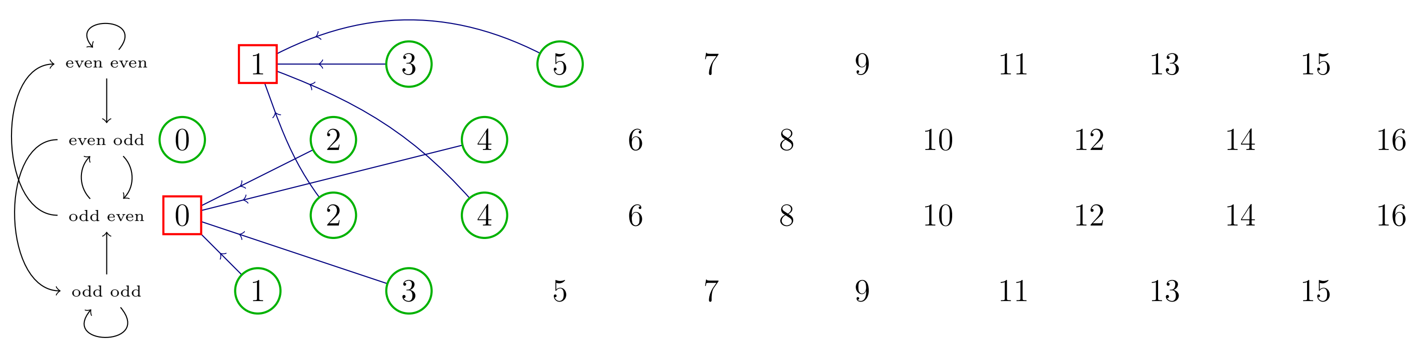
We record this swapping on the left of the diagram. We can also now fill out every position which may send to our first losing position:



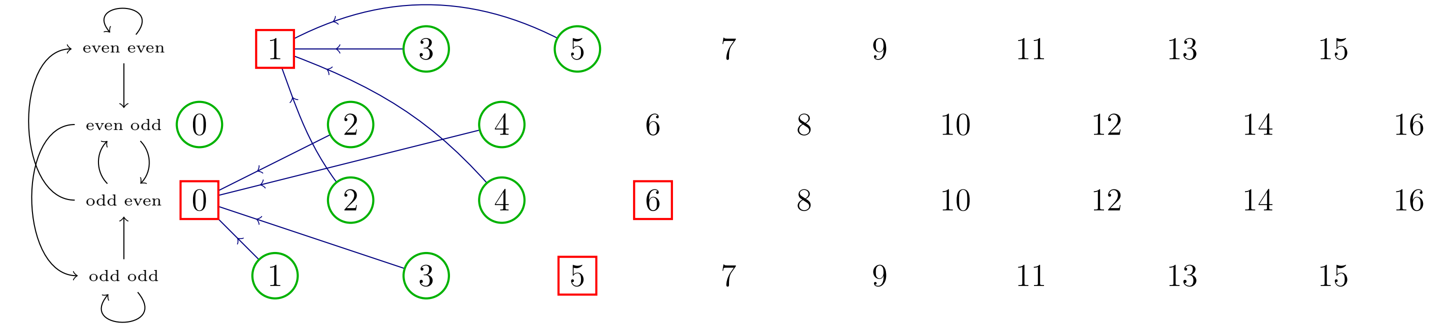
It is also not hard to see that the top left position is surely a losing one, as all it can do is send to a winning position, so we record that data as well:



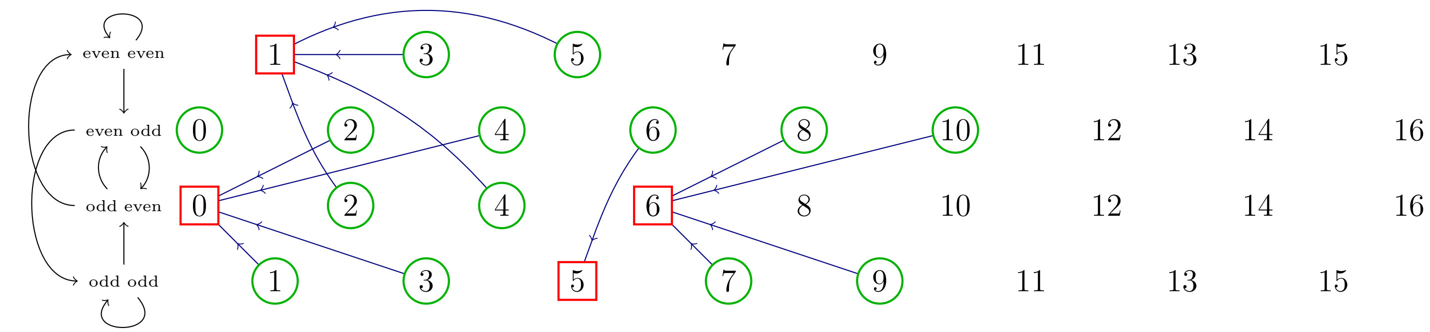
Now we search for all positions which can send to this new losing position, and mark them off as winning positions:



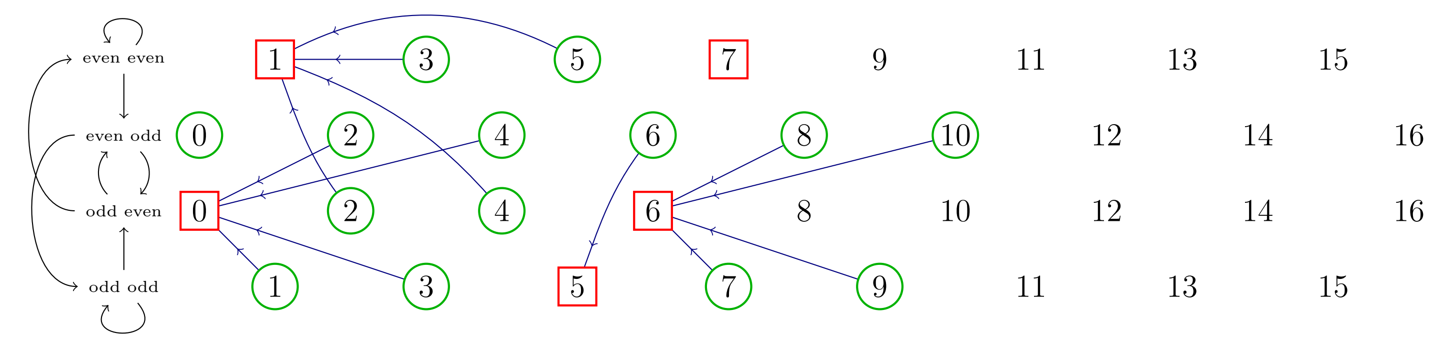
Now we search for a position which can only send to a winning one. It is not hard to see that "odd odd 5" and "odd even 6" are losing positions. We mark them off:



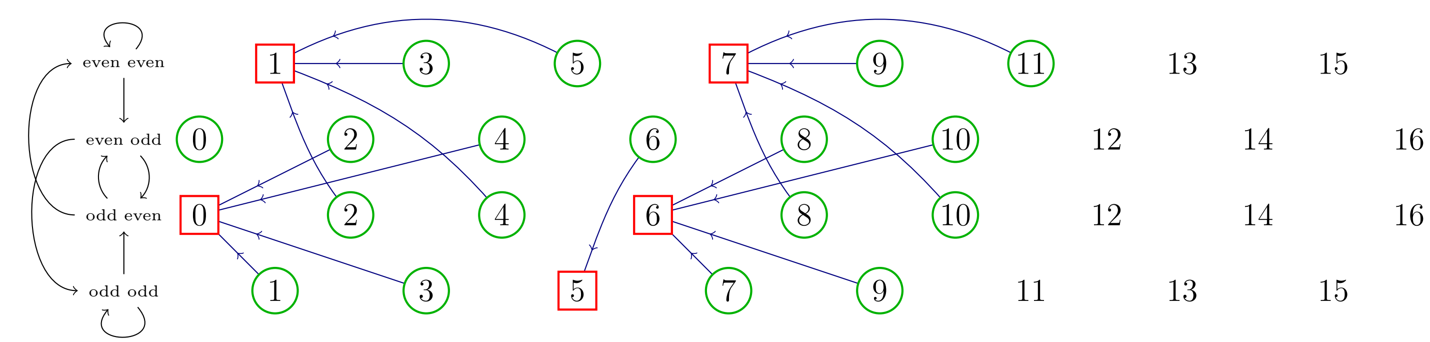
Now we look for every position which which can send to one of our losing positions. We find that "odd odd 7,9" and "even odd 6,8" can send to "odd odd 5", and "even odd 10" can send to "odd even 6". We record these as losing positions:



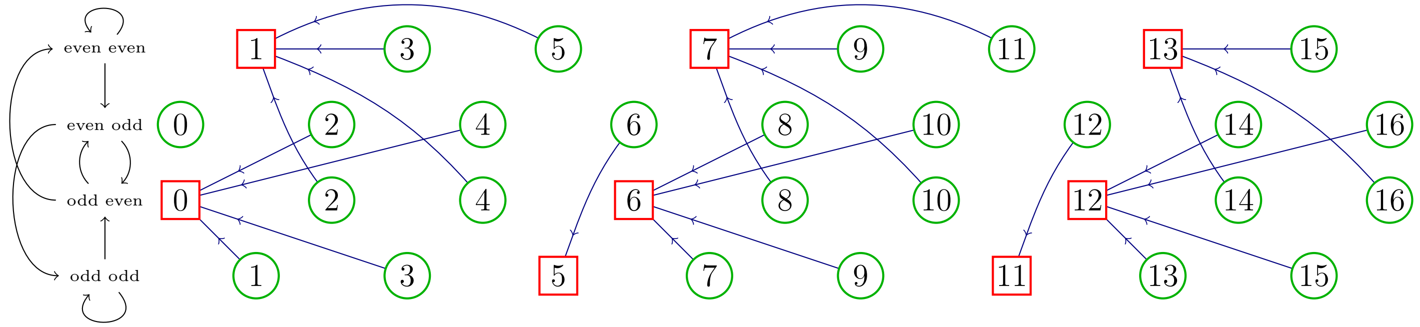
Now we see that "even even 7" can only send to winning positions, so it must be a losing one:



We record every position which can send to "even even 7" as a winning position:

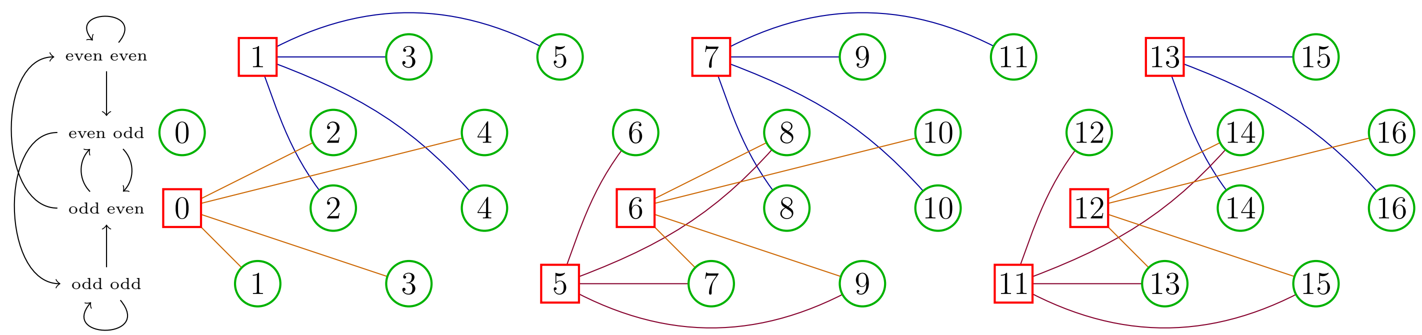


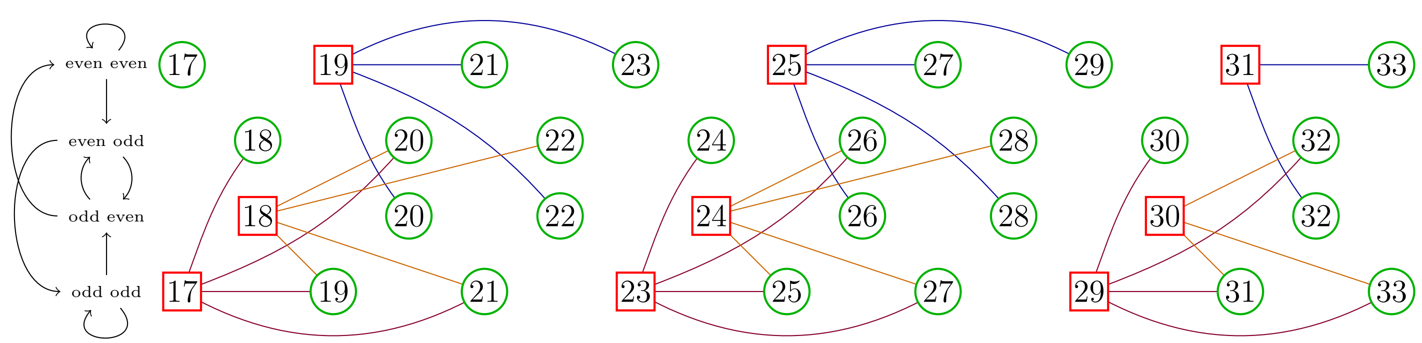
By continuing this process we can fill out the rest of the chart:



Do you see the pattern? The losing positions are the ones which are congruent to 1 (mod 6) where both players have even, congruent to -1 (mod 6) where both players have odd, and congruent to 0 (mod 6) where the first player has odd and second player has even.

You may have noticed there are other valid moves that send to losing positions. Here are all the winning moves up to 33 counters. We can easily see that the original game is a 1st player win (even even 27), and the only optimal move is to take 2. We use slightly different colours to indicate if the move is sending to 1 (mod 6), 0 (mod 6), or -1 (mod 6).





(note: this diagram is missing the line from even even 17 to even even 13).

We summarize (perhaps more succinctly) the winning strategy below, assuming you are currently in a winning position.

- If you both have even $\Longrightarrow$ send to 1 (mod 6).
- If you have odd, they even $\Longrightarrow$ send to 1 (mod 6).
- Otherwise, $\Longrightarrow$ send to 0 (mod 6) or -1 (mod 6).

This is Problem 286 in [2]. You can also consider the misère version of the game where the one ending with an odd number of counters is the winner. Perhaps more interesting is to consider what happens when we vary the number of counters being taken (for example, 1, 2, 3, 4 or 5).

References

[1] Berlekamp and Conway and Guy, *Winning Ways for your Mathematical Plays*, A K Peters, 2001.

[2] Boris Kordemsky, *The Moscow Puzzles: 359 Mathematical Recreations*, Dover, 1972.

Catenate

1. to connect, in a series of links or ties; to chain. 2. A card resource management and trick-taking game with the goal of creating sequences of cards. © Jay Schindler, 2022

BACKGROUND: I wanted to develop a card game that:
1. Focuses less on taking tricks, and more on using cards as resources
2. Doesn't have low and high value cards: each card is equally powerful yet limited
3. Can be played by 3 to 8 people with one or two decks of normal playing cards
4. Allows players to spend points to try to gain higher initiative when trading cards.
5. Is a quick game, so it can be played many times at a sitting with scores accumulating over multiple games.

GAME SUMMARY: For 3 rounds players are given 4 cards and must decide which card to put in their RESERVE, which card to put up for AUCTION, which card to use for TRADE, and which card to save for their FUTURE. They may then spend points to try to gain initiative while trading for cards. Card trading ensues. For the 4th and final round players use their FUTURE cards. Finally, players score their 13 cards by creating short and long sequences of cards to score the most points.

BEGIN WITH THE END IN MIND-- END OF GAME SCORING: By the end of the game players will have 13 cards. The goal is to create one or more numerical sequences of cards (concatenations) that score the most points. However, card suit (and suit color) is important too. There are 3 types of sequences:

Sequence in Same Suit: **3♥ 4♥ 5♥ 6♥ 7♥**
Sequence in Same Color: **10♣ J♠ Q♠ K♣ A♣ 2♠**
Sequence (no suit or color similarity): **K♠ A♥ 2♣ 3♦ 4♦ 5♠ 6♣**

The LENGTH of each sequence determines its value, but a sequence in the Same Color or Same Suit is more valuable. Note: Sequences can wrap from King to Ace-- there is a circular order to the cards.

Point Value of Sequences:

# of Cards	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Sequence (plain)	1	2	3	8	10	12	21	24	27	40	44	48	65
Sequence in color	1	3	6	10	15	21	28	36	45	55	66	78	91
Sequence in suit	1	4	9	16	25	36	49	64	81	100	121	144	169

Scoring Examples: If you have the following 13 cards at the end of game play:
J♠ Q♦ Q♣ A♠ 2♠ 3♥ 4♥ 4♦ 5♥ 6♥ 6♦ 7♦ 8♦

You could score the cards with the following sequences, for a total of 30 points.
A♠ 2♠ 3♥ 4♥ 5♥ 6♥ 7♦ 8♦ for 24 points (Sequence)
J♠ Q♣ for 3 points (Sequence in color)
Q♦ 4♦ 6♦ for 1 point each (Single cards as sequences)

Or, you could score the same 13 cards in different sequences, for a total of 34 points
A♠ 2♠ for 4 points (Sequence in suit)
3♥ 4♥ 5♥ 6♥ for 16 points (Sequence in suit)
6♦ 7♦ 8♦ for 9 points (Sequence in suit)
J♠ Q♣ for 3 points (Sequence in color)
Q♦ 4♦ for 1 point each (Single cards as sequences)

You can decide how to create your sequences, and each sequence can be scored using the most appropriate scale. (For your Final Score, you will also add the number of coins in your cup at game end-- but more on this later).

INITIAL GAME SETUP: Use one full suit of 13 cards (A through K) for each player in the game. With 3 players you might select spades, hearts, and clubs. For 5 players you would use 2 decks and could choose spades, hearts, clubs, diamonds, and spades. Try to keep suits and suit colors as balanced as possible, and let players know what all the suits are in the game deck before beginning the game.

Choose the first dealer any way you like. The dealer shuffles all the cards and deals one card face down to each player-- a GIFT card. Players must then choose the role for the card: RESERVE, TRADE, or AUCTION, and position the card. (See the discussion of these card roles and positions below.)

DEAL CARDS TO PLAYERS: The dealer deals 4 cards to each player, 2 at a time. Each player must deploy exactly one card into each of the 4 following roles. Position the cards in front of themselves on the table or playing area according to the following layout:

		AUCTION CARD(S)	
<u>PLAYER</u>: <u>LAYOUT</u>	RESERVE CARD(S)		FUTURE CARD(S)
		TRADE CARD(S)	

RESERVE CARD ROLE: One card is placed face down to the left. This card is your resource alone and will be yours until the end of the game. Use it to help create a valuable sequence at game end. Keep it face down until scoring at game end.

TRADE CARD ROLE: One card is placed face down near you. You will use this card to trade for (and win) a card available for auction. It starts as a hidden card, but will be turned face up later.

AUCTION CARD ROLE: One card is placed face down further away from you. You are offering this card up for auction in trade for another card. You may get a more helpful card during trading. It will be turned face up later.

FUTURE CARD ROLE: One card is placed face down to your right. This card will be yours to use in the last round-- the future 4th round of the game. You will have 3 cards in this position by then.

Once all players have placed all their cards, players turn their AUCTION and TRADE cards face up. Now it's time for the Dance of the Cups to determine player initiative for the upcoming trade action.

DANCE OF THE CUPS: DECIDING PLAYER INITIATIVE FOR TRADE

INITIAL SETUP: In the center of the table or game area, place a paper with a large arrow printed on it. On the arrow, line up a sequence of opaque cups (or goblets), each cup marked to identify its owner. (I use miniature plastic cups which each person has decorated with stickers of their own choosing.) Cups are mixed randomly before placing them in the line. Then, starting with the first cup at the head of the arrow, and working down to the last cup, put coins (e.g., dimes) in each cup as follows: Cup 1: 1 coin, Cup 2: 2 coins, Cup 3: 3 coins, Cup 4: 4 coins, etc., until all cups have coins in them.

DANCING THE CUPS: Place a marker next to the first cup. (I use a small miniature of a knight.) The owner of that cup may (or not) spend 1 or more coins to move along the line—one position for each coin they spend from their cup. For each position they move, they must put one coin into the cup they pass. Thus, they share their wealth to change position (and initiative). Going earlier or later in order can be strategic! Players are limited how far they can move based on the number of coins in their cup (e.g., if a cup has 2 coins in it, that player may move at most 2 spaces up or down the line of cups).

Once the person owning the first cup is done moving their cup, move the marker (knight) down to the 2nd cup in the line. The owner of the 2nd cup may also move their cup up or down the line of cups by spending one coin per position. Move the marker (knight) to the 3rd cup. That cup's owner may now move and spend coins in the same manner. This process continues down the cup positions to the end of the line. As a result, the first may be last, and the last may be first.

TRADE: Starting with the player who owns the first cup (highest initiative), and moving down the line of cups to the player with the last cup (lowest initiative), each player may now make 1 Trade.

On each player's turn, the player takes 1 of the cards in their TRADE area and trades it with a card in one player's AUCTION area. The player can also choose not to trade and keep their cards as is. The player can also choose to trade their own TRADE card with a card in their own AUCTION area.

When trading, the cards traded go into each player's hands. Each player (the trader and recipient) now decides WHERE to put the traded card: into their RESERVE, AUCTION, or TRADE area.

WINNING A CARD DURING TRADE: Cards have a cyclical order to determine which card can successfully take another card during trading. Each card can take (or win) another card of the same value or a value up to 6 positions less than its own. In other words:

An A	can take:	A, K, Q, J, 10, 9, 8	A 7	can take:	7, 6, 5, 4, 3, 2, A
A K	can take:	K, Q, J, 10, 9, 8, 7	A 6	can take:	6, 5, 4, 3, 2, A, K
A Q	can take:	Q, J, 10, 9, 8, 7, 6	A 5	can take:	5, 4, 3, 2, A, K, Q
A J	can take:	J, 10, 9, 8, 7, 6, 5	A 4	can take:	4, 3, 2, A, K, Q, J
A 10	can take:	10, 9, 8, 7, 6, 5, 4	A 3	can take:	3, 2, A, K, Q, J, 10
A 9	can take:	9, 8, 7, 6, 5, 4, 3	A 2	can take:	2, A, K, Q, J, 10, 9
An 8	can take:	8, 7, 6, 5, 4, 3, 2			

It might help to remember (and display) the following card pair guides when conducting trading:

A↘8 K↘7 Q↘6 J↘5 10↘4 9↘3 8↘2 7↘A 6↘K 5↘Q 4↘J 3↘10 2↘9

Once every player has made a trade (or chose not to), the round is over. The cups stay where they ended, and the next round begins with dealing out 4 more cards to each player. This happens for 2 additional rounds.

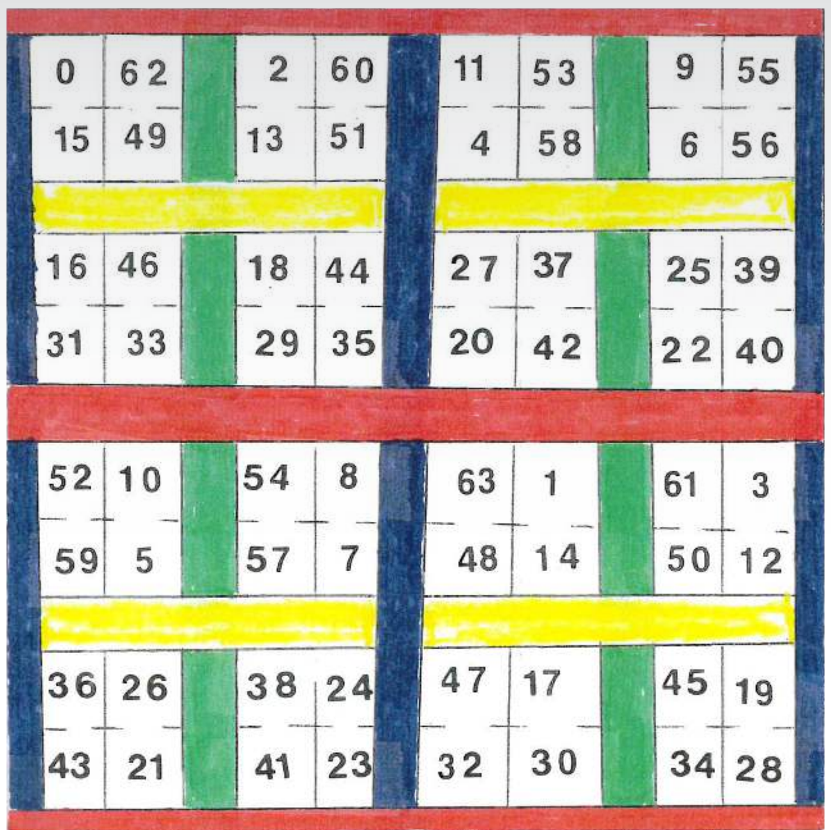
THE LAST ROUND: After 3 rounds of dealing 4 cards out to each player, there should be no cards left to deal! (Remember, you gave each player 1 Gift card at the beginning of the game.) For the last round (4th), pick up and play your 3 FUTURE cards. Place one card into each of your RESERVE, TRADE, and AUCTION areas (and none into the FUTURE area). Play out the remainder of this last round as usual.

END OF GAME: Pick up the cards from your RESERVE, any AUCTION cards remaining before you, and all TRADE cards you still have. There should be 13 cards in total. Using the Scoring Guide provided earlier, create sequences for scoring. Add the points from all of your sequences. To that total, add the number of coins in your player's cup to create your Final Score! The player with the highest Final Score wins the game. If there is a tie, the person with the longest sequence of any kind wins. If there is still a tie, the tied players share the victory. If the players decided to play multiple games before totalling the score, proceed on to the next game.

OPTIONAL RULES FOR PLAY: (I'm working on these.)
1. Add in the Joker cards, and add a Joker automa player to increase trading options and create longer sequences.
2. Allow team play. Allow table talk. When scoring, let team players combine cards and choose their best 13 cards.
3. Allow trade negotiations and contracts between players for more complex card trading.

FEEDBACK?: Have any suggestions or feedback? Do you want my rules to the Optional Rules for Play?
Please contact me (Jay Schindler) at jayvs2@comcast.net. This game is still a work in progress!

MAGIC



Some New Magic with "Most-Perfect" Magic Squares | Jeremiah Farrell | Page 58

Some New Magic With "Most-Perfect" Magic Squares

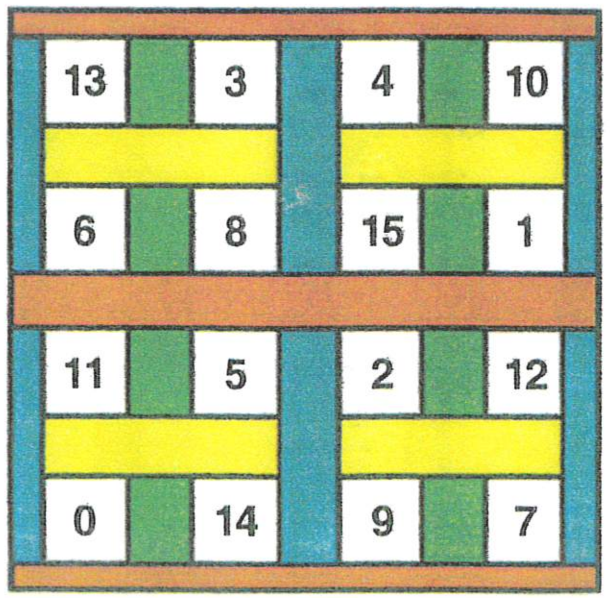
By Jeremiah Farrell

In 1998 in his article "Magic Squares Cornered" appearing in NATURE (1) Martin Gardner reports

> Dame Kathleen Ollerenshaw, one of England's national treasures, has solved a long-standing, extremely difficult problem involving the construction and enumeration of a certain type of magic square. The solution comes in a book written with David Brée. (2)

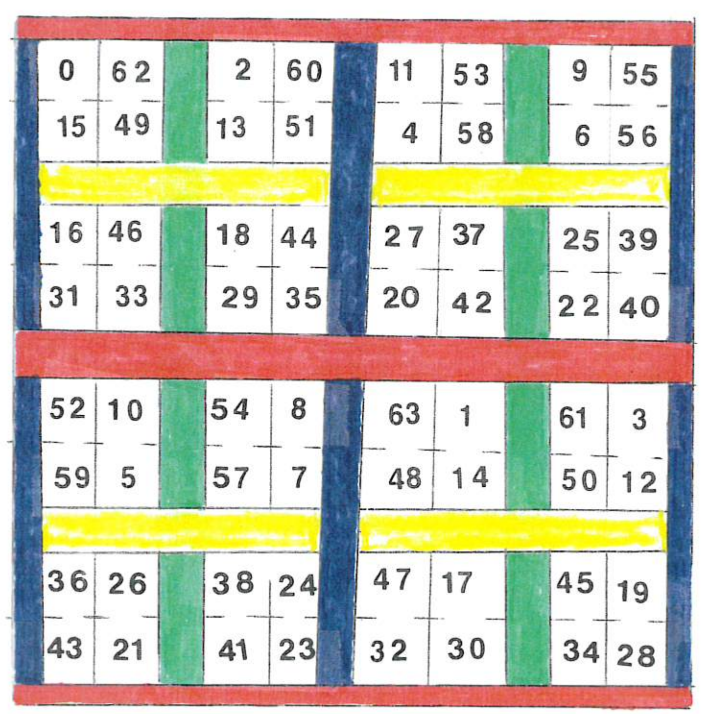
Dame Kathleen Ollerenshaw (1912-2014)
D.B.E., D.S.U., D.L., C. Math.

Most-perfect squares of order n x n have three properties. One, they are pandiagonal which means every row, column and ALL diagonals, including the broken ones sum to the same constant. Secondly, every 2x2 sub-square must sum to this constant and thirdly every pair on a diagonal ½n apart sum to half the constant. These properties force the order n to be 4m, m=1, 2,. . . . Hence the first example is of order 4x4 and the second of 8x8. Using the 16 numbers 0, 1,…,15 for n=4 and the 64 numbers 0,1,…,63 for n=8 yields these two examples among others. The constant in either is $2(n^2-1)$ and the diagonal hop sums to n^2-1.



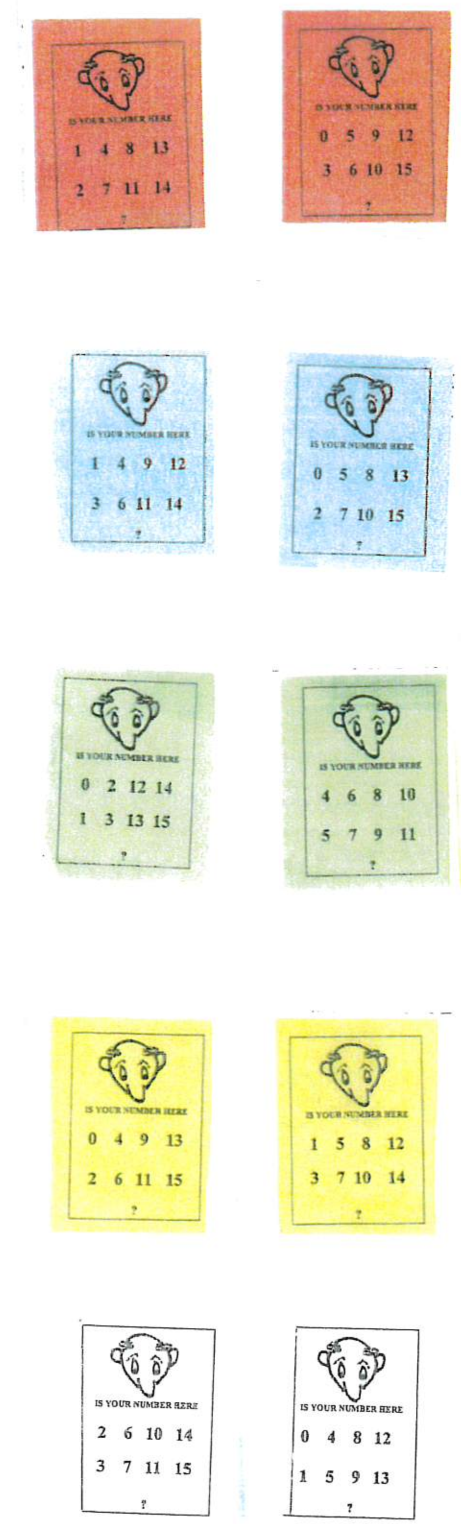
Constant = 30

Diagonal hop sums to 15



Constant = 126

Diagonal hop sums to 63



Our new magic performed on the 4x4 square starts with noting that the square can be regarded as a torus by bending the top red around to join the bottom red and joining the left and right blues to complete the doughnut shape. We supply five colored cards with all the numbers shown on either front or back. Now no matter how the cards are actually turned there will always be exactly one number showing on all five cards or not showing on all five cards. This number is the "Key" and can be changed to another by flipping the cards. As the cards lie here the Key=5.

The subject is to privately choose one of the 15 numbers and secretly choose to tell the truth to all questions or to lie to all questions. Once the subject's answers are given the magician looks quickly at the most-perfect square and correctly names the selection.

METHOD: The magician knows the Key, here 5, and traces the Yes (or No) response from the Key. The white card denotes a diagonal hop. For example suppose the left sides are showing and 4 is chosen. Telling the truth yields red=yes, blue=yes, and yellow=yes. The magician starts at the Key=5, jumps, say, yellow to 14 then red to 3 and blue to 4. Note that if the subject lies the yeses would be green to 11 and white a hop to 4.

The next page depicts a common two dimensional drawing of a four dimensional hypercube along with the "most-perfect" magic square. The parts of the 4-cube are given by the generating function $y = (1+2x)^n$. For n=4 this gives

$$y=1+8x+24x^2+32\,x^3+16\,x^4$$

Thus there are in the 4-cube 16 points, 32 lines, 24 squares, 8 cubes and 1 hypercube. All these parts may be found exactly in the magic square as follows. Turn the square into a torus and each of the 16 numbers forms a 2x2 square with each number at the bottom right. For example

7-0 or 0-14
10-13 13-3

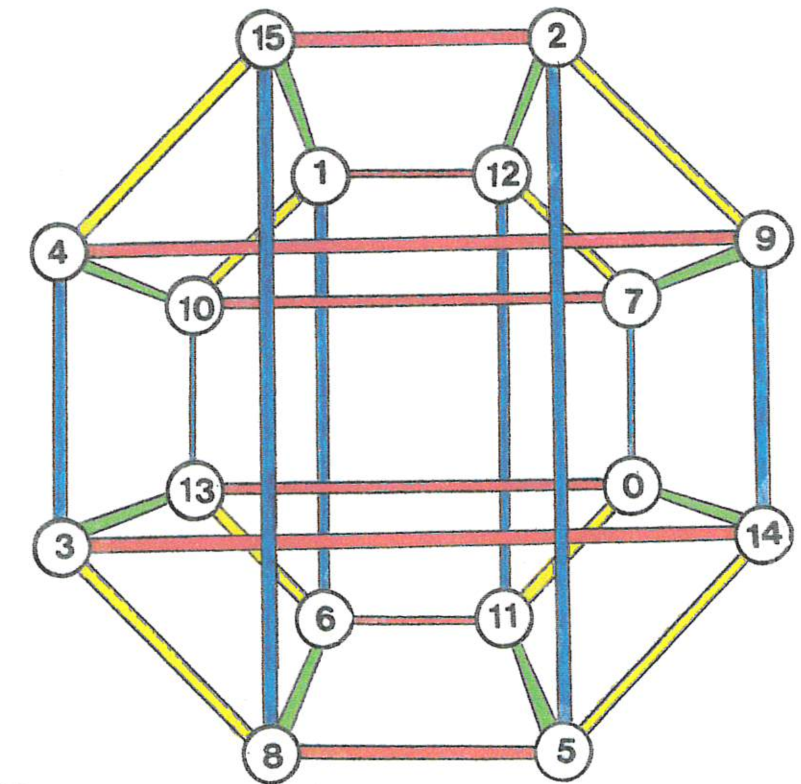
Down to 2 – 12
9 – 7

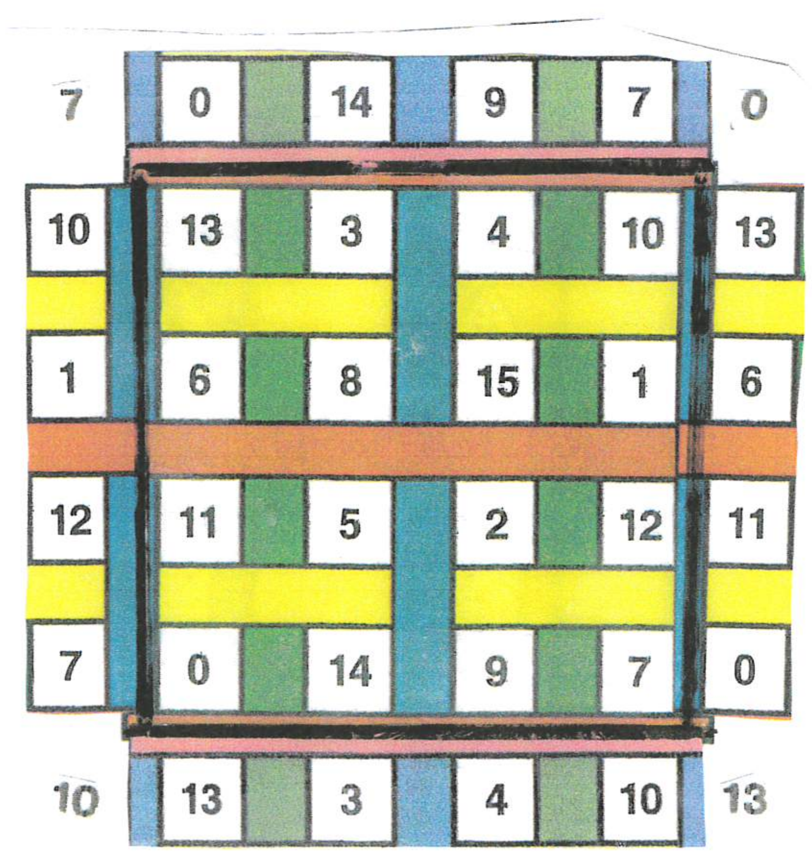
Also each of the four rows of 4 and columns of 4 yield 8 more 2x2 squares. The 8 cubes are formed from the magic square using the four double rows and the four double columns. For example

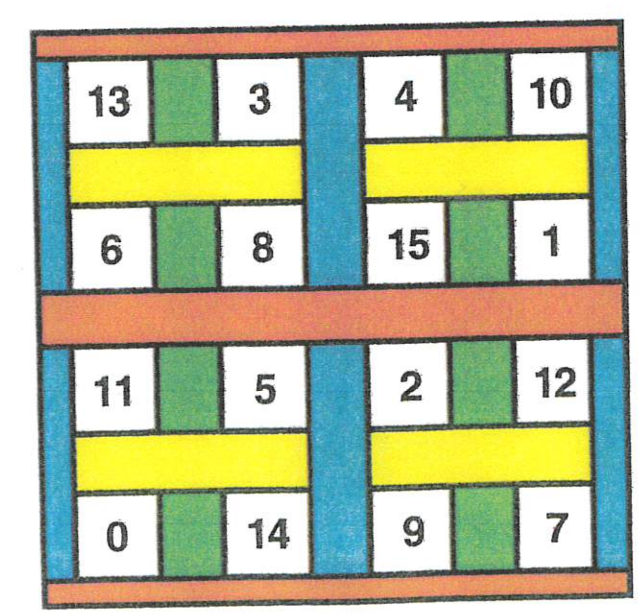
0-14-9-7 The 32 colored lines are obvious .
13-3-4-10

It may be possible to obtain other larger hypercubes from higher order most-perfect squares.

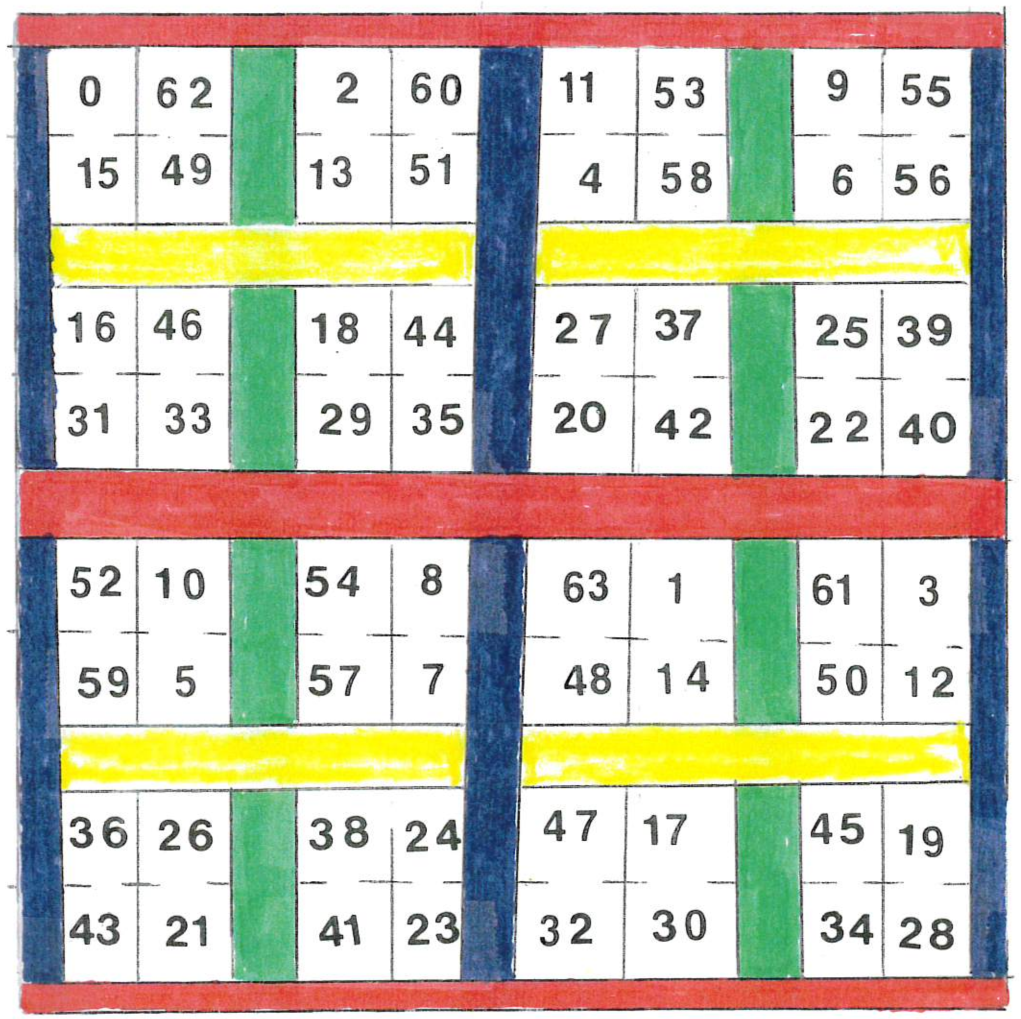
There is also an extension of the 4-cube to 5 dimensions. Notice on the magic square the unique diagonal hops of pairs that sum to 15: 0-15, 1-14, 2-13,...7-8. This turns the square into half a 5-cube. The magic can now be performed using the five cards shown. As usual no matter how the cards are turned there is always exactly one number, the key, that appears on all five. The cards displayed have key=5. When the subject secretly chooses a number and privately chooses to either lie to all cards or tell the truth to all, the magician can quickly locate the subject's selection from the magic square. For instance, suppose 8 was selected and the subject chooses to lie. In any order her responses would be with left sides up: Red No, Blue Yes, Green Yes, Yellow Yes, and White Yes. From the key=5 we trace 5 blue to 2 green to 12 yellow to 7 hop to 8.







To perform similar magic on an 8x8 most-perfect square we start by using the following square and the five colored cards on red, blue, green, yellow and white.



These cards each have half of the 64 numbers 0, 1, 2, . . . 63 on in special ways and towards the end of our trick we will ask the questions to be answered truthfully by the subject. (1) Is your number even? (2) Is your number 32 or greater?

The colored cards are

0	18	34	48
1	19	35	49
4	22	38	52
5	23	39	53
10	24	40	58
11	25	41	59
14	28	44	62
15	29	45	63

2	16	32	50
3	17	33	51
4	22	38	52
5	23	39	53
10	24	40	58
11	25	41	59
12	30	46	60
13	31	47	61

0	20	36	48
1	21	37	49
2	22	38	50
3	23	39	51
12	24	40	60
13	25	41	61
14	26	42	62
15	27	43	63

0	16	32	48
1	17	33	49
4	20	36	52
5	21	37	53
10	26	42	58
11	27	43	59
14	30	46	62
15	31	47	63

1	16	33	48
2	19	34	51
5	20	37	52
6	23	38	55
9	24	41	56
10	27	42	59
13	28	45	60
14	31	46	63

The subject is to secretly choose a number from the 64 numbers and separate the five colored cards into two piles, one with the choice on each and the other without the choice on them. Then the subject tells correctly the answer to the two questions. By looking at the most-perfect square the magician correctly names the choice.

METHOD. The 8x8 is turned into a torus similar to the 4x4 case. Either pile of colored cards will locate the correct 2x2 square containing the subject's choice on starting at the Key 54-8.
57-7

The white card is a single diagram hop. When the answers to the two questions are known the choice is identified.

As an example suppose 37 is the choice. One pile of colored cards contains the green, yellow and white cards and the other pile red and blue. Using either pile the magician finds the current 2x2. From the Key 54-8 trace green to 52-10,
57-7 59-5
yellow to 36-26 and then white
43-21
A hop to 27-37, and locates
20-42
the choice 37 once the two questions are answered.

The reader will note that this magic is similar but much harder to fathom than the old chestnut trick using base two cards.

Martin Gardner remarks that the authors for the first time were able to find all the most-perfect squares of all orders. For example, not counting reflections or rotations there are 48 4x4s and 368640 8x8s. When you reach 36x36 the number is 2.76754×10^{44} - around a thousand times the number of pico-pico-seconds since the Big Bang. Gardner adds

> This solution of one of the most frustrating problems in magic-square theory is an achievement that would have been remarkable for a mathematician of any age. In Dame Kathleen's case it is even more remarkable, because she was 85 when she and Brée finally proved the conjectures she had earlier made. In her own words, "The manner in which each successive application of the properties of the binomial coefficients that characterize the Pascal triangle led to the solution will always remain one of the most magical mathematical revelations that I have been fortunate enough to experience. That this should have been afforded to someone who had, with a few exceptions, been out of active mathematics research for over 40 years will, I hope, encourage others. The delight of discovery is not a privilege reserved soley for the young."

Perhaps the reader would prefer an alternative to the two questions that must be answered truthfully in the 8x8 case. This can be accomplished by using the following two orange cards instead.

0	16	36	52
1	17	37	53
2	18	38	54
3	19	39	55
8	24	44	60
9	25	45	61
10	26	46	62
11	27	47	63

0	16	32	48
2	18	34	50
4	20	36	52
6	22	38	54
9	25	41	57
11	27	43	59
13	29	45	61
15	31	47	63

They are to be added to the two piles of five cards under the same provisos. The key 2x2 with 7 as the lower right is transposed by either pile into another pile (unless one of the four members of the key is chosen) and the magician traces the new 2x2 from the lower right across the oranges with the solid or dashed sides. That is, the four entries of the 2x2 will be, starting at the lower right entry as follows. Neither orange stays on lower right, both oranges cross the dashed and solid lines, and one of the oranges goes across only the dashed or solid lines. If that new entry occurs on any of the colors that is the subject's choice. If not, then this is the "not" pile and the subject's choice is the diagonally opposite choice.

For example suppose the subject chooses 39. The two piles will then be first red blue, green, dashed and second yellow, white, solid. If the magician traces the first pile it goes from key 7 to 39 and from the second pile it would lead to 22 but 22 does not appear on either yellow, white or solid so this is the "not" pile and the subject's choice is the diagonal 39 instead.

In summary, the red, blue, green, yellow and white cards identify the proper 2x2. Then the oranges locate one of the four entries starting at the lower right. If the pile is noted to be the "hit" pile that entry is the choice. If the pile is noted to be the "not" pile, the diagonal is the choice.

REFERENCES

(1) Magic Squares Cornered", Martin Gardner, NATURE, vol. 395, 17 September 1998.
(2) *Most Perfect Pandigonal Magic Squares, Their Construction and Enumeration*, by Kathleen Ollerenshaw and David Brée. ISBN 0 90501 06X. From The Institute of Mathematics and Its Applications, Catherine Richards house, 16 Nelson Street, Southend-on-Sea, Essex, SS1 1EF U.K.

November 1996
Hendersonville, NC
Martin and Charlotte Gardner
with Karen Farrell

November 1996
Hendersonville, NC
Martin and Charlotte Gardner
with Jeremiah Farrell

Martin Gardner and I in Norman, OK, 2004

MARTIN MAGIC FOR G4G14

by Lacey Echols and Jeremiah Farrell

The magician asks the subject to place the following nine words in a 3x3 grid so that each row and column anagrams into MARTIN. These nine are supplied on discs. An, At, In, Ir (iridium), It, Mn (manganese), Mr, Mt (meitnerium), Ra (radium). A correctly formed grid will be what Leonard Euler called "semimagic".

The subject is now instructed to turn the disc over and to carefully interchange any two rows or any two columns as often as they wish. Of course the interchanges are such that each row or column will still spell MARTIN. Suppose the 3x3 has finally become, for example, as in Figure 1.

The magician says "If you were to give me two discs in, say, a row I would be able to easily name the third in the row."

"Instead", he asks, "I want you to choose any three discs in the grid but to make it hard on me make sure they are each from a different row and column – that is, no two in the same row or the same column." The subject chooses three such and places one face down and hands the magician the remaining two. The magician then immediately and correctly names the face down third disc.

METHOD. The chosen three will always have exactly one letter in common and all the others different. In fact the nine entries can be represented in base three arithmetic where A=0, I=1, M=2, and T=0, R=3, N=6. This makes the pairs become the grid in Figure 2. This suggests that the magic can be done with these numerical discs as well.

EXAMPLE. Supposes the subject selects from the words in Figure 1 the three Mt, Mr, Mn so that M is the only common. With numbers these would be 02, 32 and 62 with 2 common. The magician has memorized the associations and can after practice quickly name the missing disc. For MARTIN the magician memorizes AIM and NRT. That is, the common letter is noted and the opposite set of three are the matches.

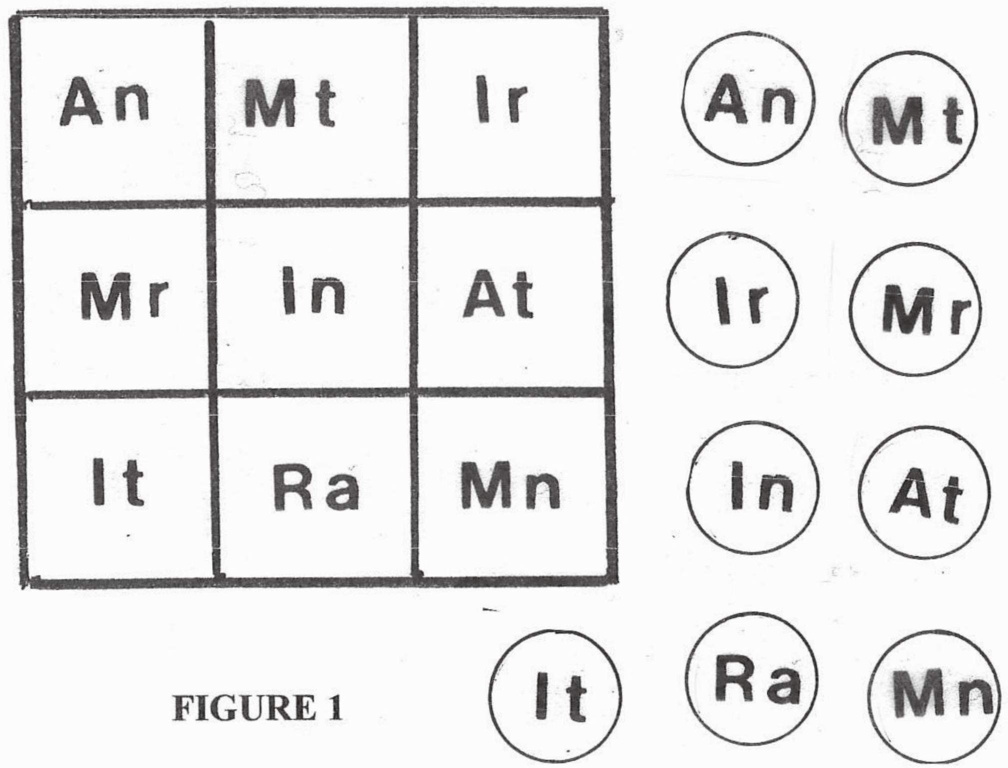


FIGURE 1

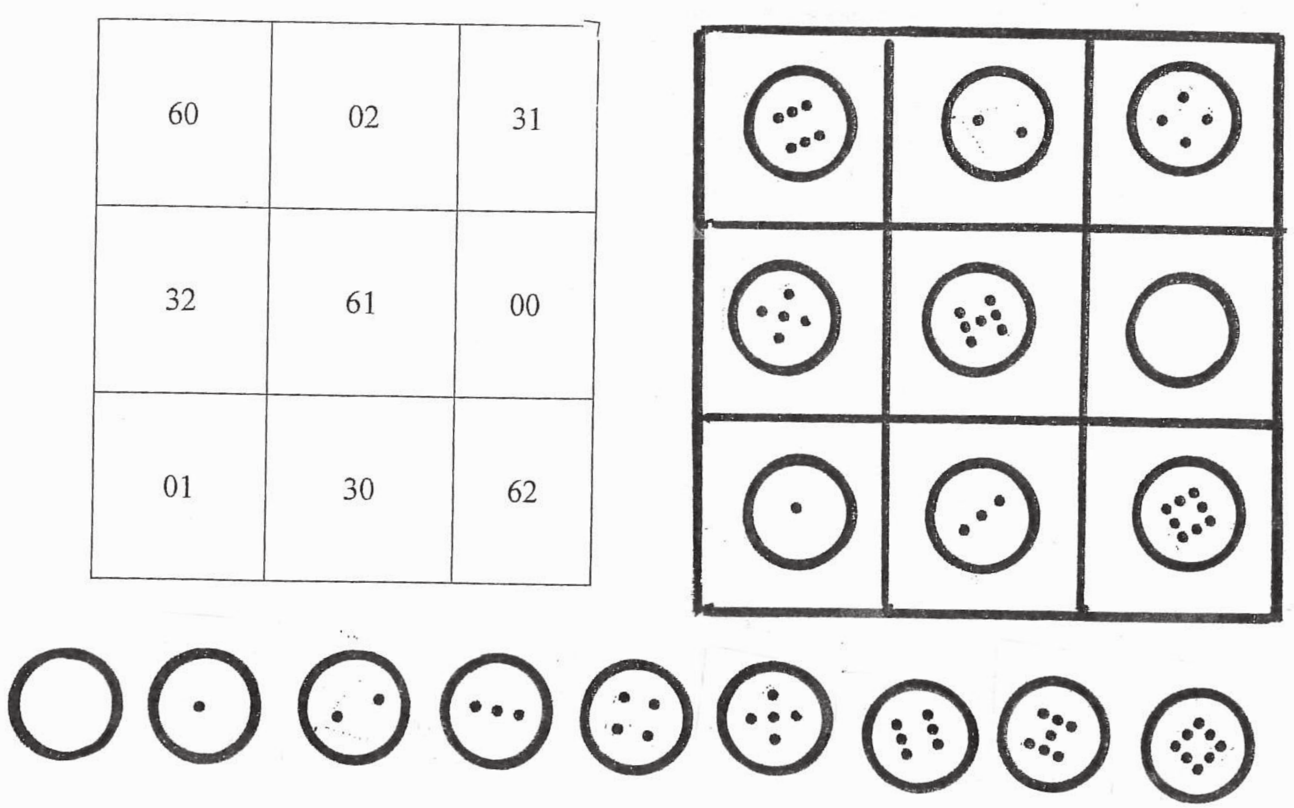


FIGURE 2

G4G14 Exchange Gift

In 1994 I worked on a collaboration project with Martin...

Binary Arts wanted to produce "Martin Gardner Presents 6 Mathematically Based Magic Prediction Tricks" and I spent several months corresponding with him trying to assemble a worthy collection.

Although the project never made it to market, Martin shared a number of wonderful paper-based mathematical magic tricks with me and we built a great working friendship, something I will always treasure.

Over the years I have periodically looked back through this material for inspiration, but somehow until last month I never noticed one envelope which had been tucked into a neglected folder.

Sifting for G4G exchange gift inspiration, immediately when I saw the trick I realized this would be a perfect contribution. Not just the mathematical simplicity, also the "typewritten with edits-by-pen" style that was characteristic of Martin's correspondence.

Then I noticed something curious... the date. Martin wrote this letter on October 21, 1994, his 80th birthday! How fitting to think of him spending a little time that day formulating a simple mathematical magic trick for the world. And how fun, nearly 28 years later, to unearth it and be able to share with all of you.

Happy Belated Birthday everyone!

Bill Ritchie
ThinkFun co-founder
With colleagues Melinda Contreras and Sophie Miller

To make your own tokens, print this page and Martin Gardner's letter on the next page, double-sided, with long-edge binding. Then, cut out all four squares below.

7	5	4	1

G4G14 Exchange Gift

MARTIN GARDNER
110 Glenbrook Drive
Hendersonville,
N.C. 28739
(704) 693-3810

21 Oct 94

Dear Bill:

I just read a British book on magic that describes a simple and clever force of the number 18 that was new to me.

It uses the four cardboard disks (or squares) numbered as shown on the enclosed set.

Show the audience that the disks bear numbers 1 through 8. Pretend to mix them up but leave them on the table so that the odd didigts (1,3,5,7) are on top.

Turn your back and ask him to turn over any two disks, then total the four numbers showing. They will always be 18.

The trick works the same if you start with the four even digits showing.

The following occurred to me. Let the spectator begin by turning the disks at random so that the numbers on top are genuinely randomized. If he leaves the disks with all even or all odd showing, you go at once into the presentation.

If only one disk is the wrong parity, explain that while your back is turned he is to turn over two disks, "like so." To illustrate, turn over the "wrong" disk.

If two disks are wrong, then of course you illustrate what he is to do by turning over ~~the~~ two two disks.

This dodge automatically insures that the four disks are all the same parity, but leaves the impression that they have been randomized.

All best,

Martin

To make your own tokens, print this page and Martin Gardner's letter on the next page, double-sided, with long-edge binding. Then, cut out all four squares below.

2	3	6	8

A Ternary Hamming Code in a Magic Trick

Ricardo Teixeira

April 2022

For the Ternary Hamming Code trick, a magician is able to identify a number from 1 to 80 that a volunteer chose even if he decides to lie about one piece of information. It involves an error detection and correction algorithm, called *Hamming Code* (Hamming, 1950).

This ternary trick expands the binary version published on last Gathering for Gardner Exchange Book (Teixeira, 2018), where the volunteer could only pick a number from 1 to 15. Tricks with Hamming Code are common on the literature, for instance (Ehrenborg, 2006; Mateer, 2013; Teixeira, 2017), but this is the very first trick using a ternary version of Hamming Code. This trick was first presented during the 2018 MOVES conference, and the following description resembles (Teixeira & Park, 2020).

The Trick

Description: A volunteer thinks of a number from 1 to 80, he also selects a color from the rainbow (7 options). Then, the magician shows 7 colored cards with several numbers for the volunteer to say whether he sees his number or not, the volunteer lies on the card of his chosen color. The magician is able to find on which color the volunteer had lied, and then tell the chosen number.

Material: Copy and cut cards on appendix, if you have crayons you could color the cards accordingly. You could use actual fidget spinners of cut the ones on the appendix.

Preparation: Put the cards in order (red, orange, yellow, green, blue, magenta, purple). Practice how to check the options for ternary matching (see instructions below).

Performance: Gisele, the magician, will read Arthur's mind.

1. Gisele asks Arthur to pick a number between 1 and 80, and one of the colors of the rainbow (red, orange, yellow, green, blue, indigo or violet);
2. Gisele explains that Arthur has to say whether he can see his chosen number on each of the cards she shows. If the number appears, he has to say which color the number has on the card (black or red);
3. But Gisele also explains that Arthur should tell a lie on the card having the color he had chosen. The lie could be of any type: lying whether the number is on the card or not, or even lying about the color that the number has on that card;
4. For every time he says "red" for a card, Gisele lays the *fidget spinner* (see appendix) with its red circle pointing up. If Arthur says the number is "black" on a certain card, Gisele lays the *fidget spinner* with its black circle pointing up.

Finally, if he says that number is not on the card, she lays the *fidget spinner* with its white circle pointing up;

5. She arranges the *fidget spinners* side-by-side from left to right;
6. Once all seven cards are dealt, she looks at the *fidget spinners* and she can tell in which color the lie happened, which type of lie was told, and which number was selected.

Trick: Trick is based on the ternary extension of the Hamming Code. The first four cards resemble a ternary-digit trick with numbers 1 to 80. Having the color *red* means the correspondent position has digit 1, color *black* means that position has digit 2, while if the number does not appear (fidget spinner has white circle pointing up), then the correspondent position would be zero. If there were no lies allowed, then we'd only need the first four cards. Simply, we would add the top left number on each color (red or black) for the corresponding cards in which the volunteer claims to see the number.

Summary:

- Don't see the number (color is white) = 0.
- The number is red = 1.
- The number is black = 2.

However, a lie was told and we are also trying to discover where the lie happened, and which type of lie it was.

We use the following system: if no lie was told, then the result would be that the 7 positions would satisfy the following relationships.

- $(\text{position}_2 + \text{position}_3 + \text{position}_4) = \text{position}_5 \pmod 3$
- $(\text{position}_1 + \text{position}_3 + \text{position}_4) = \text{position}_6 \pmod 3$
- $(\text{position}_1 + \text{position}_2 + \text{position}_4) = \text{position}_7 \pmod 3$

The first four cards will serve to compute the chosen number by determining what is the ternary expansion of the chosen number.

The last three cards are the "checking digits".

During the trick, for each of the checking positions (positions 5, 6 and 7), the magician needs to mentally compute the *discrepancy*:

- $Discrepancy_5 = (\text{position}_2 + \text{position}_3 + \text{position}_4) - \text{position}_5 \pmod 3$
- $Discrepancy_6 = (\text{position}_1 + \text{position}_3 + \text{position}_4) - \text{position}_6 \pmod 3$
- $Discrepancy_7 = (\text{position}_1 + \text{position}_2 + \text{position}_4) - \text{position}_7 \pmod 3$

And while displaying the *fidget spinners* for each position, if a discrepancy is detected, then:

- If discrepancy is 1: let the *fidget spinners* be displayed a little higher than others (in such way that the magician can see, but it would not call the audience's attention);
- If discrepancy is 2: let the *fidget spinners* be displayed a little lower than others.

According to the number of discrepancies, the position of the lie can be determined by similar analysis as the binary case (**?**). Once the position of the lie is identified, the sum of the value of the discrepancies and the value represented by the lie (the color of the *fidget spinner*: white=0, red=1, black=2) will identify the type of lie.

To find the *position* of the lie:

There are discrepancies on:	Then lie was on:
5	5
6	6
7	7
5 and 6	3
5 and 7	2
6 and 7	1
5, 6 and 7	4

Once the position of the lie is known:

- If there was only one discrepancy: the corresponding card is the lie, and the actual value of the card is supposed to the value of the *discrepancy* added to the *lie-value* (the value that corresponds to the *fidget spinner*'s color) (mod 3).
- If there were more than one discrepancy, then they all have the same value (1 or 2):
  - the value of each discrepancy and the lie-value adds up to three: then the person lied about the color; otherwise
    - if the lie-value is not zero: the true value is zero;
    - if the lie-value is zero: the true value is "3 minus discrepancy".

Explanation: It is based on the theory developed in the exercises.
Hint: Practice the error recognition. At first, it may take you a while to figure out the position of the lie and its type. Onde identified, fix the lie, before telling the chosen number.

Examples

Example 1: Suppose that the chosen number is 37, and the chosen color is magenta (the sixth card).

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Red	Yes, Red	No	Yes, Red	Yes, Black	Yes, Red (lie)	No
1	1	0	1	2	1	0

- $Discrepancy_5 = (\text{position}_2 + \text{position}_3 + \text{position}_4) - \text{position}_5 \pmod 3 = 1 + 0 + 1 - 2 = 0$ (no discrepancy)
- $Discrepancy_6 = (\text{position}_1 + \text{position}_3 + \text{position}_4) - \text{position}_6 \pmod 3 = 1 + 0 + 1 - 1 = 1$ (discrepancy)
- $Discrepancy_7 = (\text{position}_1 + \text{position}_2 + \text{position}_4) - \text{position}_7 \pmod 3 = 1 + 1 + 1 - 0 = 3 = 0 \pmod 3$ (no discrepancy)

Since, there is only one discrepancy, then that's where the lie is. The color of that card was supposed to be $1+1$ (the actual value plus the value of the discrepancy), hence color 2 (black). The chosen number is $1 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 1 \times 3^0 = 27 + 9 + 0 + 1 = 37$.

Example 2: Suppose that the chosen number is 70, and the chosen color is orange.

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Black	No (lie)	Yes, Black	Yes, Red	Yes, Red	Yes, Black	Yes, Red
2	0	2	1	1	2	1

- *Discrepancy*$_5 = (\text{position}_2 + \text{position}_3 + \text{position}_4) - \text{position}_5 \pmod 3 = 0 + 2 + 1 - 1 = 2$ (discrepancy)
- *Discrepancy*$_6 = (\text{position}_1 + \text{position}_3 + \text{position}_4) - \text{position}_6 \pmod 3 = 2 + 2 + 1 - 2 = 3 = 0 \pmod 3$ (no discrepancy)
- *Discrepancy*$_7 = (\text{position}_1 + \text{position}_2 + \text{position}_4) - \text{position}_7 \pmod 3 = 2 + 0 + 1 - 1 = 2$ (discrepancy)

Since, checking digits 1 and 3 show discrepancy, the lie is on the second card (orange). Since he told 0, the correct value was "3 minus discrepancy": $3 - 2 = 1$ (red). The chosen number is $2 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 54 + 9 + 6 + 1 = 70$.

Example 3: Suppose that the chosen number is 16, and the chosen color is red (the first card).

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Black (lie)	Yes, Red	Yes, Black	Yes, Red	Yes, Red	No	Yes, Black
2	1	2	1	1	0	2

- *Discrepancy*$_5 = (\text{position}_2 + \text{position}_3 + \text{position}_4) - \text{position}_5 \pmod 3 = 1 + 2 + 1 - 1 = 3 = 0 \pmod 3$ (no discrepancy)
- *Discrepancy*$_6 = (\text{position}_1 + \text{position}_3 + \text{position}_4) - \text{position}_6 \pmod 3 = 2 + 2 + 1 - 0 = 5 = 2 \pmod 3$ (discrepancy)
- *Discrepancy*$_7 = (\text{position}_1 + \text{position}_2 + \text{position}_4) - \text{position}_7 \pmod 3 = 2 + 1 + 1 - 2 = 2$ (discrepancy)

Since, checking digits 2 and 3 show discrepancy, the lie is on the first card (red). Since the value of the discrepancy is 2 and he told 2, the correct value was "3 minus discrepancy": $2 - 2 = 0$ (white). The chosen number is $0 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 0 + 9 + 6 + 1 = 16$.

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Appendix

Ternary Hamming Code Cards

27	28	29	30	31	32
33	34	35	36	37	38
39	40	41	42	43	44
45	46	47	48	49	50
51	52	53	54	55	56
57	58	59	60	61	62
63	64	65	66	67	68
69	70	71	72	73	74
75	76	77	78	79	80

9	10	11	12	13	14
15	16	17	18	19	20
21	22	23	24	25	26
36	37	38	39	40	41
42	43	44	45	46	47
48	49	50	51	52	53
63	64	65	66	67	68
69	70	71	72	73	74
75	76	77	78	79	80

3	4	5	6	7	8
12	13	14	15	16	17
21	22	23	24	25	26
30	31	32	33	34	35
39	40	41	42	43	44
48	49	50	51	52	53
57	58	59	60	61	62
66	67	68	69	70	71
75	76	77	78	79	80

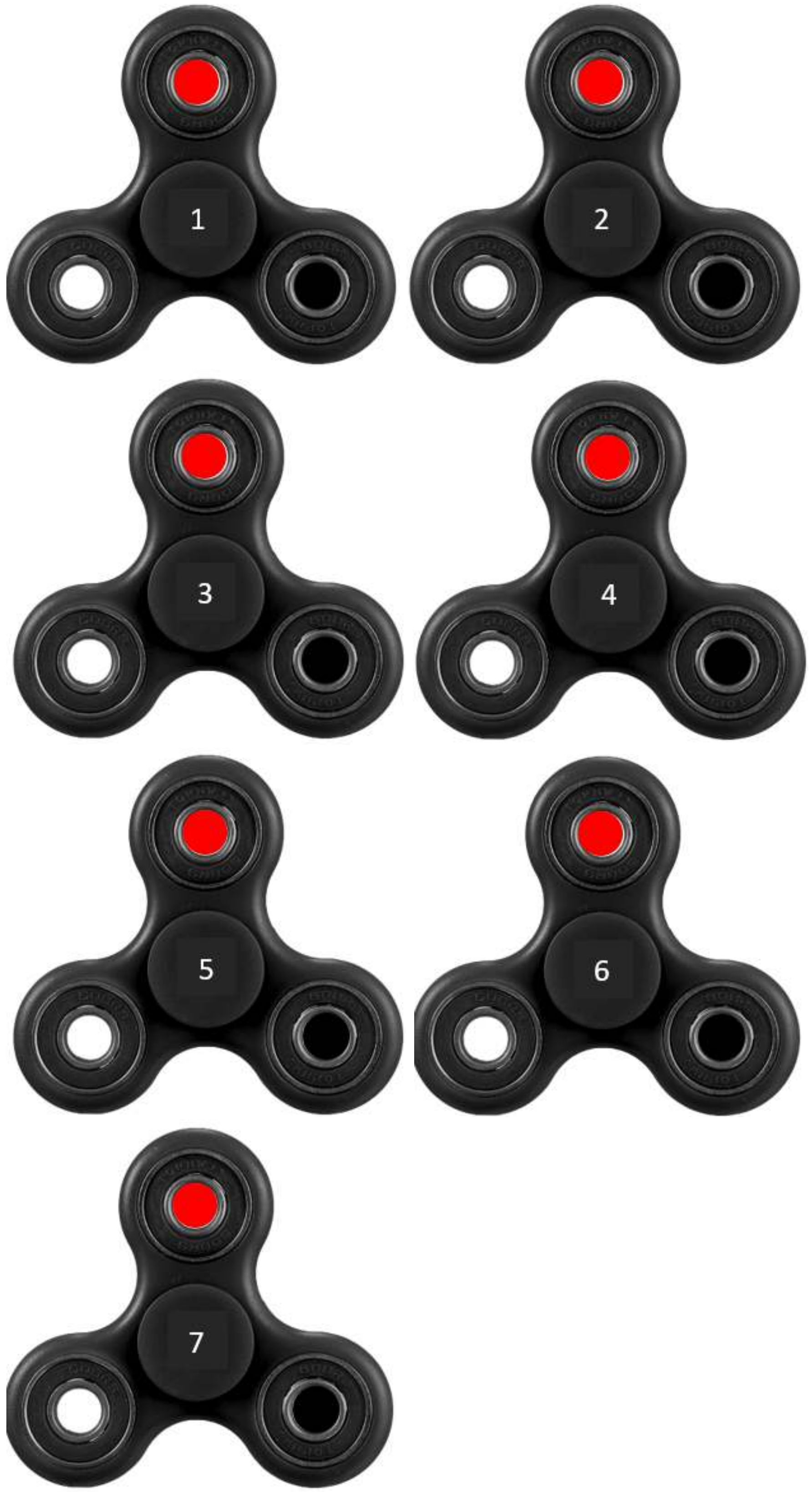
1	2	4	5	7	8
10	11	13	14	16	17
19	20	22	23	25	26
28	29	31	32	34	35
37	38	40	41	43	44
46	47	49	50	52	53
55	56	58	59	61	62
64	65	67	68	70	71
73	74	76	77	79	80

1	2	3	4	6	8
9	10	12	14	16	17
18	20	22	23	24	25
28	29	30	31	33	35
36	37	39	41	43	44
45	47	49	50	51	52
55	56	57	58	60	62
63	64	66	68	70	71
72	74	76	77	78	79

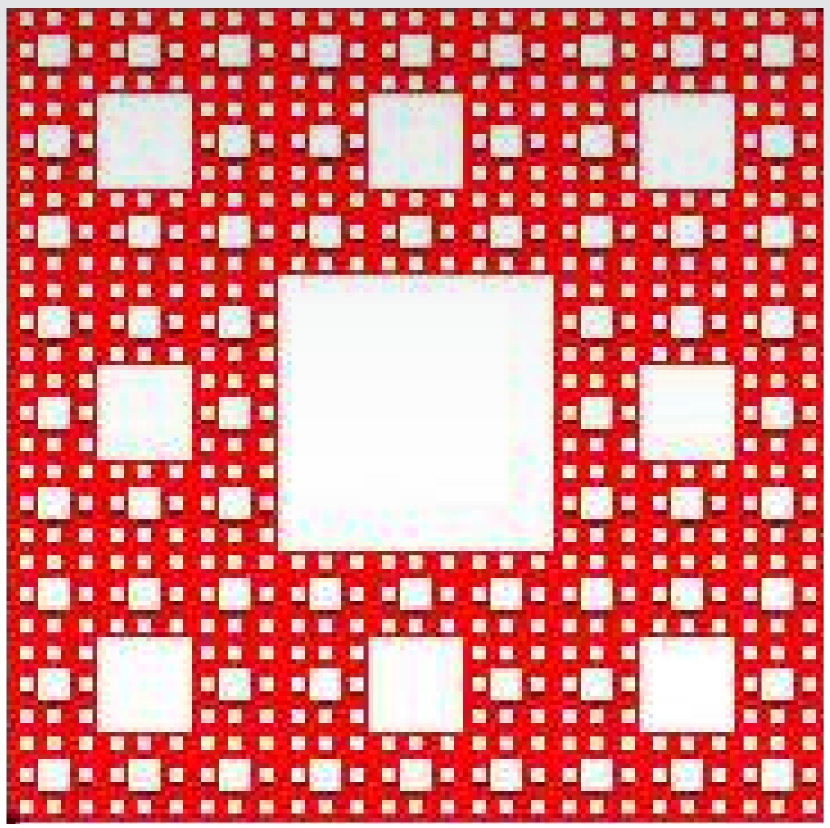
1	2	3	4	6	8
10	11	12	13	15	17
19	20	21	22	24	26
27	28	30	32	34	35
36	37	39	41	43	44
45	46	48	50	52	53
54	56	58	59	60	61
63	65	67	68	69	70
72	74	76	77	78	79

1	2	4	5	7	8
9	10	12	13	15	16
18	20	21	23	24	26
27	28	30	31	33	34
36	38	39	41	42	44
46	47	49	50	52	53
54	56	57	59	60	62
64	65	67	68	70	71
72	73	75	76	78	79

Ternary Hamming *Fidget Spinners*



MATH



Traveling Through the Sierpinski Carpet & Menger Sponge | Derek Smith | Page 186

The Law of the Third

Adam Atkinson (ghira@mistral.co.uk), G4G14, April 2022

The Law of the Third may be my second-favourite gambling myth. It's a distant second, since it's not remotely as good as the Samaritani Formula (described at a previous G4G), but it seems to turn up in more countries and languages, which means there's more chance G4G attendees may run into it or its victims. In French it's called “la loi du tiers”, in German “Zwei-Drittel-Gesetz” (where the name has "two thirds" in it) and in Italian it's "la legge del terzo". I'd love to hear about sightings in other countries and languages.

It's a reasonably interesting example of pseudomathematics. I run into it, or am asked about it, at intervals of several years, and have been since my first encounter with the Samaritani Formula 20+ years ago. Sometime people ask me how the Law of the Third works (It doesn't!), or where the trick is, or they are incredulous that I think the Samaritani Formula is a myth and say “Next you'll be telling me the Law of the Third is a myth!”. Yes, I suppose I shall!

What exactly does the Law of the Third say? Well, this isn't entirely clear since the people who spread it don't like to be, or are incapable of being, terribly clear or specific.

You will probably find two main variants of the LOTT.

(i) If you draw one item from n, n times, $n/3$ items will fail to appear.

(ii) If you draw one item from n, n times, $n/3$ items will fail to appear, $n/3$ will appear once, $n/3$ will appear twice.

If someone says one of these in exactly these terms, they are implicitly saying that draws are not independent. So your die / roulette wheel etc. must contain sensors, memory, gyroscopes, motors. Nanotechnology or magic may be involved.

And it will be claimed that this either is a method for making money playing roulette (or lotteries, or similar), or that it is a mathematical principle which allows one to find a method for etc. etc. To be fair, this is the bit that's a myth. Without too much effort, we can state a version of the LOTT which is “true but irrelevant” in that it's a true statement that does NOT allow us to win money playing roulette. In much the same way that the Samaritani Formula can be expressed in a way which is true but irrelevant.

“$n/3$ items will” may be stated as exactly $n/3$, or at most, at least, approximately, on average, … "at most" "at least" and "approximately" are all false in any reasonable model of the universe. "often approximately" and "on average" are more or less true but irrelevant.

How do they say you can make money using this? They seem not to want to be very clear about this. Perhaps so they can sell you worthless software, books or consultancy services, or so if you can't make it work they can claim you weren't using it properly. The details don't really matter, since it can't possibly work (in any plausible model of the universe). It's a bit like someone saying they've found two odd integers whose sum is odd. No, they have not. There's no point checking particular examples: the

Tales of Wild Dice

J. Buhler, A. Gamst, A. Hales*

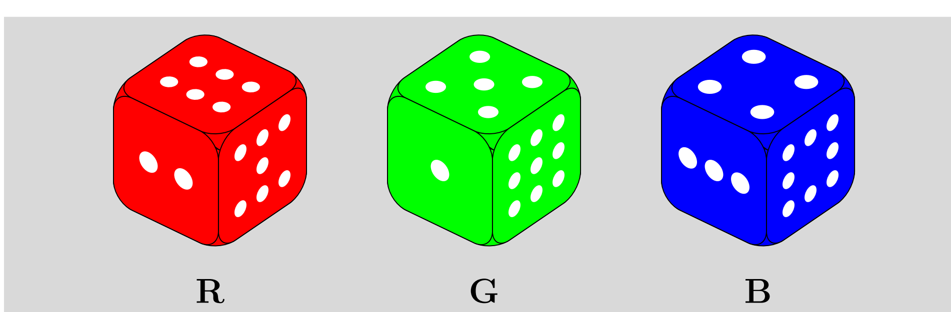
Martin Gardner played a large role in popularizing nontransitive dice, starting with his December, 1970, column in which he focused on dice due to Efron. Another example, due to Moon and Moser [7], is pictured below. For us, dice are lists (with repetition allowed) of equally likely integer outcomes of a "roll," and we write the Moon/Moser dice as

$$\mathbf{R} = [2, 6, 7], \quad \mathbf{G} = [1, 5, 9], \quad \mathbf{B} = [3, 4, 8].$$

(If you wonder what's on the hidden faces, then we'd tell you that each face is identical to its antipodal face, so that we should have written, e.g., $\mathbf{R} = [2^2, 6^2, 7^2]$; however, this die is equivalent to the simpler form above.)

In addition to being nontransitive (**R** is higher than **G**, on average, **G** is higher than **B**, and yet **B** is higher than **R**), these dice have an even stranger property: the triple $\mathbf{R}^{[2]}$, $\mathbf{G}^{[2]}$, $\mathbf{B}^{[2]}$ is also a nontransitive cycle, but in the opposite direction! (The superscript on $\mathbf{R}^{[2]}$ indicates the die whose roll is the sum of two independent rolls of **R**.) To our knowledge, these curious facts were first noticed by Tom Leighton around 1990, when he was working on notes for a course at MIT (for an amusing discussion of swindles based on these dice, see section 17.3 in the book [6] that grew out of these notes). This idea also appeared in a paper by Allen Schwenk [8], whose title, *Beware Geeks bearing Grifts*, is hard to beat.

Ron Graham asked how far these curious properties could be pushed. He showed that much more exotic outcomes were possible, and his effervescent (and, OK, insistent) personality led to two joint papers: one [3] showing how to produce arbitrary tournaments, in a sense that will be made precise below, and another [2] that shows how to fix a fascinating gap that arises when one tries to argue that these examples are actually explicit. This is an overview of some of the results and techniques, and our real goal is to entice you into reading those papers! We are deeply grateful for Ron's mathematical and non-mathematical friendship, his insights, and the extraordinary opportunity to collaborate with him.



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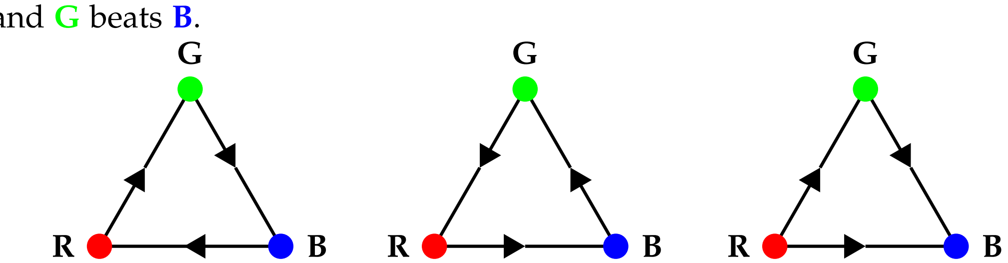
Dice

For us, a *die* (plural: dice) is a finite list of equally likely outcomes (rolls). Dice can be added or multiplied by constants; e.g.,

$$\begin{aligned} \mathbf{B} + \mathbf{R} &= [5, 6, 9, 10^3, 11, 14, 15], & 2\mathbf{G} &= [2, 10, 18] \\ \mathbf{B} - \mathbf{R} &= [-4, -3^2, -2, 1^2, 2^2, 6], & \mathbf{G}^{[2]} &= [2, 6^2, 10^3, 14^2, 18]. \end{aligned}$$

Note that multiplying a roll by 2 is a very different from adding 2 independent rolls. Bracketed exponents indicate repeated addition, and exponents on the values indicate multiplicity (repetition).

Given a set of k dice, we are interested in the results of comparing all pairs of the above dice when each is rolled n times and the results are added. For the **RGB** triple of dice we want to know the 3 pairwise comparisons from the set $\{\mathbf{R}^{[n]}, \mathbf{G}^{[n]}, \mathbf{B}^{[n]}\}$, for all $n \geq 1$. The results are summarized in the figure below for $n \leq 100$. For each n, we label the edges of a triangle whose vertices are the dice with an arrow that points from the winner to the loser. Three different "tournament results" occur. For a few small n, they are nontransitive cycles; for a few other small n, and apparently all $n \geq 9$, the result is the same as for $n = 3$, i.e., **R** beats both **G** and **B**, and and **G** beats **B**.



$n = 1, 4$ $\qquad$ $n = 2, 5, 8$ $\qquad$ $n = 3, 6, 7, 9, \ldots, 100$

This means, for $n = 1$, that (loosely speaking) **R** is "better" than **G**, **G** is better than **B** and yet **B** is better than **R**. This is easy to check; e.g., looking at the list for $\mathbf{B} - \mathbf{R}$ above, we see that 5 of the 9 outcomes have **B** winning, so **B** will beat **R** more often than not. For the **RGB** dice, nontransitivity for $n = 1$ is perhaps a bit surprising, but the fact that for $(\mathbf{RGB})^{[2]}$ case (i.e., $\mathbf{R}^{[2]}, \mathbf{G}^{[2]}, \mathbf{B}^{[2]}$) the tournament is a nontransitive cycle in the reverse direction is doubly surprising, and the fact that $(\mathbf{RGB})^{[3]}$ is neither of the nontransitive cycles is perhaps triply surprising. For the nth powers (as we will call them) for $n \geq 9$ the outcome is always the same tournament, mentioned above. As Schwenk's title hints, these properties offer numerous opportunities for swindles (a.k.a. grifts), at least if wagers, n, and the gradual emergence of the full situation are carefully managed.

Our primary goal in the next section is to construct sets of k dice $D = \{D_1, \ldots, D_k\}$ that exhibit vastly more deranged properties.

For the **RGB** dice, all of the pairwise comparisons seem fair because the means (averages) of the compared dice are equal. Indeed, if the means were unequal, well-known results from Statistics 101 show that, at least when larger powers are taken, dice with higher means will win, in the long run. In other words, for the sake of finding counterintuitive examples it suffices to consider only sets of dice that all have the same mean.

There is an interesting higher order grift that might arise if the players are mathematicians. Imagine that you and an opponent play this game repeatedly (perhaps for the sake of speed and magnifying the small winning margins as n increases, letting a computer roll the dice, of course using a fair random number generator). You spar over who chooses the first die, and what n will be, and after a while you both understand what's going on, e.g., understand the above figure whether or not you cop to that knowledge to your opponent. The size of the bets has been steadily increasing.

At one point, your opponent offers you the following game. She names an n, you pick whether to chose first or second, and then, rather than the grubby rolling of dice and doing arithmetic, or even simulating that using a computer, it is agreed that whoever chooses the first die will win if they can give a reasonably concise proof that they win (in the probabilistic sense of having a greater probability of rolling a higher number). If you've taken Probability 301 then you know that the computation comes down to comparing the median of the difference of nth powers of two of the **RGB** dice to the mean, which is 0. As n gets large, the median and mean are close (by the Central Limit Theorem), but their difference is governed by the "skewness" of the distribution (roughly, which way it leans away from being a symmetric normal distribution). This primarily depends on the third moment $\sum \mathbf{Pr}(X = x)x^3$ where X is the difference of the nth powers of two of the dice. Moreover, you dimly recall that the winning margin goes to 0 as a function of n something like $c/\sqrt{n}$. Making the choice, and giving a proof, is trivial for $n \leq 8$ and is easy for large n because of the theorem below, which you will have to cite. The stakes are of course tripled, and since you know the full story and have a sure thing, you accept.

Your opponent says that n will be 10^{24}. Which, in case you are counting (and use American terminology) is one septillion. This is a bit unnerving, but of course you know to choose **R**. Your proof begins by quoting the following theorem, which you cleverly stored on your phone during the break.

Theorem 1. *Let X be a die (an integer-valued random variable) with span 1 and mean 0. Then for n going to infinity,*

$$\mathbf{Pr}(X > 0) - \mathbf{Pr}(X < 0) = \frac{c}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right), \quad c = \frac{-\mu_3(X)}{3\sqrt{2\pi\mu_2(X)}}.$$

You (tediously) continue your proof by explaining that the "little-oh" term $o(1/\sqrt{n})$ means that for all positive ε there is an $n_0 = n_0(\varepsilon)$ such that the term is less than $\varepsilon/\sqrt{n}$ for $n \geq n_0$. And that the jth central moment of a die X is

$$\mu_j(X) = \sum_x \mathbf{Pr}(X = x)\, x^j.$$

Finally, the *span* of the (integer-valued) X is the largest m such that the values of X are contained in a single congruence class $a + m\mathbf{Z}$ modulo m. Any element of the congruence class is called the *shift* of the die. For instance the values of **G**, namely, 1, 5, and 9, are all 1 mod 4 and contained in the congruence class $1 + 4\mathbf{Z}$ modulo 4; moreover, this is clearly the largest possible m with that property, since two of the rolls differ by 4, so the span of **G** is 4. The shift is only defined modulo m, so the shift could be said to be 5, 1, or -3. However, all of the pairwise differences of the three dice have values that differ by 1, so their span is 1 and you are relieved that the above theorem applies.

You finish by asserting that **R** is clearly going to win — the moments are easy to compute, and the chosen n is (insanely) large.

You are horrified when your opponent points out that this is not a proof because you haven't yet named an explicit function $n_0(\varepsilon)$ and proved that it works. As unlikely as it seems given the outcomes up to $n = 100$, you have not proven that the third moment term dominates the error term for $n = 10^{24}$.

You may now be in a spot of trouble (especially because the stakes have been increasing). Stay tuned for useful remarks in the last section.

In the meantime, you might take some solace from having noticed that there is an intuitive reason for the minus sign in front of the third moment term in the Theorem—if the third moment is positive, then there has to be more probability mass that is negative in order to balance the die so that its mean is 0. So the median will be negative.

Tournaments

Let $D = \{D_1, \ldots, D_k\}$ be a set of k dice. The result of all $k(k-1)/2$ pairwise comparisons between the k dice can be recorded as an antisymmetric matrix whose the entry in row i and column j is 1 if D_i beats D_j in the long run, 0 if the contest is fair (i.e., a probabilisitic tie), or -1 if D_j beats D_i in the long run. This matrix is a *tournament* if there are no ties, and a *partial tournament* if there are ties. For instance, the three tournaments in the figure above could be represented (less elegantly) as the following matrices:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

If X is a die then, in order to cope with ties, and reduce the outcome to ± 1 if there is a winner (and 0 for a tie), it is convenient to define the "positivity" of a die X as $w(X) = \operatorname{sgn}(\mathbf{Pr}(X > 0) - \mathbf{Pr}(X < 0))$, where $\operatorname{sgn}(x)$, for a real number x, is 1, 0, or -1 according to whether x is positive, zero, or negative.

Fundamentally, $w(X)$ measures whether the median of X is above or below 0. For a set $D = \{D_i\}$ of k dice, define the tournament on their nth powers setting the element in the ith row and jth column to be

$$T_n(D)_{ij} = w(D_i^{[n]} - D_j^{[n]}).$$

(This is only a partial tournament if some of the off-diagonal entries are 0.)

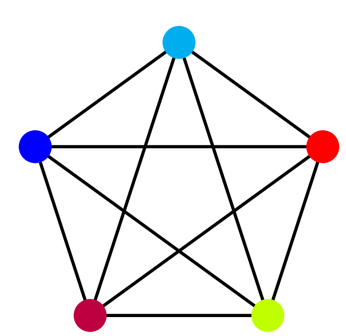
Suppose that there are $k = 3$ dice. Then there are 3 pairwise contests of interest, and $8 = 2^3$ possible ways for those contests to be decided (ignoring ties), and therefore 8 possible tournaments. Only 3 of the 8 possible tournaments showed up in the contests between the nth powers of **R**, **G**, and **B**. Wouldn't it be cool, or at least more nontransitive, if there was another set of dice A, B, C such that all 8 of the possible tournaments occurred as a "dominance tournament" on $A^{[n]}$, $B^{[n]}$, and $C^{[n]}$, for some value of n?

The next theorem says that such a set exists. Worse, it says that such a set that realizes all possible k-person tournaments exists for *any* $k > 3$. Worse yet, there will be no "limiting tournament" as in the case of the **RGB** dice (i.e., a tournament that is the result for all sufficiently n), because the dice are so exquisitely balanced that each tournament not only occurs, but occurs infinitely often.

Theorem 2. *For every $k > 2$ there is a set of k dice $\{D_i\}$ such that for any $k \times k$ tournament matrix T on k players there are infinitely many n such that $T = T_n(D)$.*

A proof can be found in [3]. For the sake of giving a sense of what is going on, we consider the example of $k = 5$ dice.

First, there are $2^{10} = 1024$ possible tournaments (!). The 10 edges of the complete graph on 5 dice (illustrated below) can be oriented in 1024 ways, and each of those tournaments on $D = \{D_i\}$ can be realized as the tournament graph on the nth powers, for infinitely many n.



This sounds like a tall order. Curiously, it turns out that our only recourse is to go back to Theorem 1 and look at the omitted case: dice with spans larger than 1. Also it turns out that the third moment term is just an annoyance, and it simplifies things to just require that the third moment is always 0.

Suppose that D_i is a collection of k dice that have mean and third moment equal to 0, with shifts a_i, spans m_i respectively. It isn't hard to show that mean and third moment of $D_i^{[n]}$ are 0, and its shift and span are na_i and and m_i. And to show that the span of a sum of two die is the gcd of their spans, and the shift of the sum is the sum of the shifts.

If x is a real number, its fractional part, written $\{x\}$, is x minus the largest integer less than or equal to x. It is convenient, for the sake of stating the improved version of Theorem 1, to define

$$\langle x \rangle = \begin{cases} 1 & \text{if } 0 < \{x\} < 1/2 \\ -1 & \text{if } 1/2 < \{x\} < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the "otherwise" case occurs precisely when x is an integer or half-integer. Carefully generalizing Theorem 1 for spans larger than 1 leads to a very concrete description of the (matrix of) tournaments $T_n(D)$ for sets of dice $\{D_i\}$ that that have $\mu_1 = \mu_3 = 0$, at least for large enough n.

Theorem 3. *Suppose that D_i are k dice with mean and third moment equal to 0, and with shifts a_i and spans m_i. Define*

$$d_{ij}^{[n]} = \frac{n(a_i - a_j)}{\gcd(m_i, m_j)}.$$

Then the ij entry of the tournament matrix $T_n(D)$ is

$$T_n(D)_{ij} = \operatorname{sgn}\left(\frac{c \langle d_{ij}^{[n]} \rangle}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right), \quad c = 1/(3\sqrt{2\pi}).$$

Note that if n is large enough the argument of sgn has the same sign as its $\langle\ \rangle$ term, unless that term is 0.

This shows that we can compute tournaments matrices for large n once we know the shifts and signs, but still begs the question of how we can specify the D_i in any straightforward and general way. The following Lemma gives a very practical answer!

Lemma 1. *Given a positive integer m and any integer a there is a die X with shift a, span m, mean 0, and third moment 0.*

If you guess to look at what you can do if X has only 3 values (with repetitions allowed), the proof is a rather unenlightening exercise in high school algebra, though it can be made more appealing by using some simple linear algebra on 2×3 matrices. We also expect that the only chance for more elegant dice (fewer repeats) is to use more values.

Via this lemma, the following table specifies five dice with mean and third moment 0, with the indicated shifts and spans.

a_i	2^4	2^3	2^2	2^1	2^0
m_i	2^5	2^9	2^{12}	2^{14}	2^{15}

Our goal is to prove Theorem 2 which, loosely speaking, asserts the existence of dice that are so wild that their powers have all possible tournaments. To do this we have to show that every tournament matrix T on 5 contestants is of the form $T_n(D)$, for the set D just described, and some positive integer n.

Once all of the results are unraveled, this comes down to showing that the 10-tuple ν of all $(a_i - a_j)/\gcd(m_i, m_j)$ from the above table satisfies the conclusion of the following Lemma, which is something of a "mod-2 equidistribution" law. In our application of the lemma, K is 10, since ν has 10 components. For the sake of stating the Lemma, we let H_K be the set of all elements x in $[0, 1]^K$ which have no coordinate x_i equal to 0, 1/2 or 1. This is a disjoint union of 2^K open cubes C of side length 1/2. The Lemma asserts that for every C the integer multiples $\{n\nu\}$ of ν in 10-space intersect some translate $u + C$ of the half-cube C by an *integer* 10-tuple u.

Lemma 2. *With the above notation, there is a vector ν such that for every open cube C in H_K there are infinitely many integers n such that $\langle n\nu \rangle$ lies in C, where $\langle u \rangle$ denotes the result of applying the bracket operator to every component of u.*

The reader might enjoy either or both of the (nontrivial!) exercises of proving the Lemma, or of showing that the 10-dimensional ν arising from the 5 dice above satisfies the Lemma.

This finishes the outline of the proof of Theorem 2.

Comments.

A number of remarks are in order.

#1: Linguistically, "intransitive" (meaning definitely not transitive) is probably a better term than "nontransitive" (meaning not necessarily transitive), but we stick with the latter as it has become thoroughly ingrained in the mathematical literature on the topic.

#2: Although there may have been related ideas that arose earlier, the specific idea of nontransitive dice seems to have first appeared in work of Steinhaus and Trybula [9] in the late 1950s. A number of interesting references to results on nontransitive dice can be found in the bibliographies of the more recent papers in the bibliography below. Also, there is a lot of information online, including, of course, the Wikipedia article on Intransitive Dice as well as web pages by James Grime, Oskar Deventer, and no doubt others.

#3: In aiming at tournaments, we skipped over the interesting case of n-cycles. The method of construction of the 5×5 matrix (extracted verbatim from [1])

7	8	9	10	25
4	5	6	23	24
2	3	20	21	22
1	16	17	18	19
11	12	13	14	15

is clear. A bit of careful thinking about the dice formed from its rows should convince you that it is easy to construct nontransitive cycles of any length. Indeed, rows of this matrix form a nontransitive cycle of length 5: each row is better than the one below it, and the bottom is better than the top! One question that has been around since the 1960s is: what is the highest possible success probability around the cycle? I.e., given n, what is the largest p such that there is a nontransitive n-cycle with $\mathbf{Pr}(D_{i+1} > D_i) \geq p$ for all i? A definitive treatment appeared in the Monthly recently [5], and it confirms that the largest p is

$$p = 1 - \frac{1}{4\cos^2(\pi/(n+2))} = \frac{3}{4} - O(1/n^2).$$

(Strictly speaking, for dice in our sense, this is the supremum of all such p, but this value can actually be attained if irrational probabilities are allowed.) In other words, the best winning probability around a cycle approaches 3/4 from below, as n goes to infinity.

#4: The dice in the proofs might not be aesthetically pleasing or "practical" since our goal was only to push the envelope on what was known to be possible. There are lots of opportunities to do better. One obvious open question is to find a reasonable set of 3 dice that realize all 8 tournaments. One measure of the size of a set of dice is the least common multiple L of the spans of its dice (though we think that this is not exactly the same as "practical"). For $n = 3$, the smallest possible value of L is 10 and, perhaps slightly surprisingly, a set of 3 dice exist with $L = 10$. For $n = 4$, easy arguments show that the optimal L satisfies $64 < L \leq 512$ (the upper bound coming from the construction above). In fact $L = 68$ is possible (and almost certainly best possible). This seems to suggest that wild dice exist more or less as soon as there is any room for them to exist.

#5: A full proof of Theorem 3 can be found in [3] where the probability in question is initially expressed as a contour integral.

#6: Explicit estimates error estimates for Theorem 3, i.e., with overt functions $n_0(\varepsilon)$ that you might need (in the circumstances mentioned above), can be found in [2]. The quest for explicit error estimates arises in number theory, probability, and elsewhere. In the case of [2] (first order lattice Edgeworth error estimates, in the lingo) common wisdom was probably that precise estimates were hard to find, likely to be uninteresting because they would be unrealistically large, and not terribly useful because computers can compute values for large n, leaving no doubt as to what actually happens in, say, dice tournaments. Although a variety of ideas were needed, the estimates in [2] turned out to be unexpectedly good at giving realistic estimates even in fairly pathological cases. Be that as it may, computations do give a good sense of what is going on, and you sort of have to be a mathematician (at least at heart) to enjoy this quest for explicit estimates.

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SEATTLE UNIVERSAL MATH MUSEUM

What is Ethnomathematics, and How Can it Help Solve a Math Problem?

3/10/2022
by Tracy Drinkwater

Synopsis: Brazilian mathematician Ubiratan D'Ambrosio coined the term "Ethnomathematics" in 1987. Decades later, we often still see math as separate from culture. How can D'Ambrosio's approach help us change how we teach students who have felt no connection between their lives and the subject of math?

I have loved math since I was a young child. I've always been drawn to puzzles, games and LEGO® sets. I took for granted that these activities were inherently mathematical, and that much of my fluency with math was due to my experiences within my own home and family culture. For most of my life, and especially as a math educator, I have been fascinated by other cultures and how they value and teach mathematics. In my work I have collected stories, videos, and articles about math from around the world and through the centuries. I've been fascinated by how mathematics has been developed throughout history.

Math is often considered a universal language and acultural. However, this can be misleading, as math has been created and discovered to serve the needs of communities. Various cultures throughout history and all over the world have developed methods of counting and measuring. Each method was unique until more and more cultures came into contact with one another through trade, migration, travel and conquest. Communication then necessitated that communities use a common vocabulary and methods. Some ideas were shared, while others were practically stolen – and false credit was given to those who first published the theorems, rather than to their original source. From the necessity for universal vocabulary to communicate cross-culturally, mathematics came to be viewed as the foundation and universal language for science, engineering and technology. In the process, it came to be viewed as separate from the humanities and culture.

As a Seattle University Instructor teaching *Math Methods* for graduate students in the Master in Teaching program, I discovered the article “What is ethnomathematics, and how can it help children in schools?” by Ubiratan D'Ambrosio[1]. I used this article to stimulate discussion in our Culturally and Linguistically Responsive Teaching (CLRT) unit. D'Ambrosio, a Brazil mathematician, explains:

> The term ethnomathematics is used to express the relationship between culture and mathematics. The term requires a dynamic interpretation because it describes concepts that are themselves neither rigid nor singular-namely, ethno and mathematics (D'Ambrosio 1987). The term ethno describes "all of the ingredients that make up the cultural identity of a group: language, codes, values, jargon, beliefs, food and dress, habits, and physical traits." Mathematics expresses a "broad view of mathematics which includes ciphering, arithmetic, classifying, ordering, inferring, and modeling" (pp. 2-3). Many educators may be unfamiliar with the term, yet a basic understanding of it allows teachers to expand their mathematical perceptions and more effectively instruct their students.

And so it seems, by having separated math from culture, our American educational system has implemented a procedural and acultural approach to teaching math. And e have failed to reach and teach many of our students, especially those of color. Below is a graph from the Seattle Times representing data for Washington State public school students from before (2019) and during (2021) the COVID-19 pandemic. The data tells the tragic story of how few students in the majority of categories are meeting or exceeding standards in mathematics.[2]

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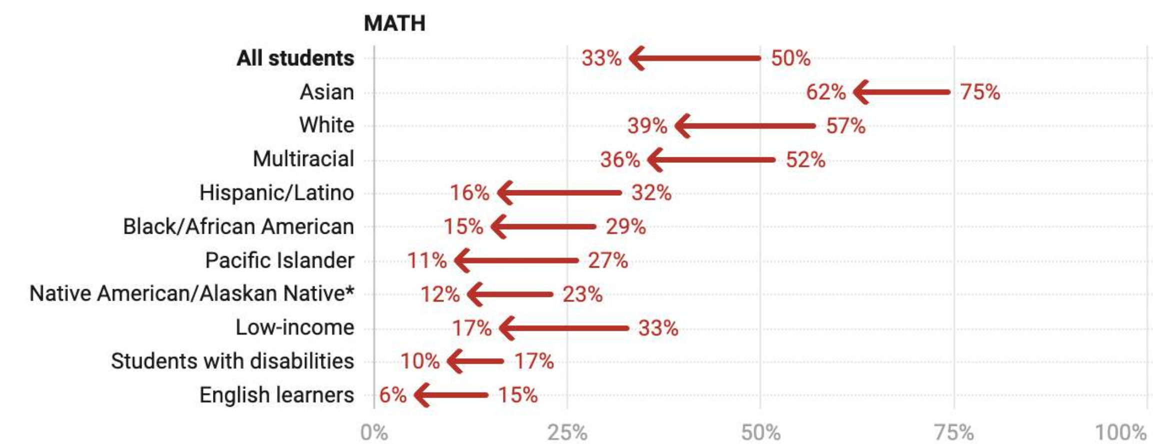
[1] D'Ambrosio, Ubiratan. “What is ethnomathematics, and how can it help children in schools?” *Teaching Children Mathematics*; Reston; Feb 2001.
https://www.researchgate.net/publication/284702127_What_is_ethnomathematics_and_how_can_it_help_children_in_schools

[2] Bazzaz, Dahlia. “Washington Students' Test Scores Drop Significantly in First Exams since Pandemic Began.” *The Seattle Times*, The Seattle Times Company, 18 Jan. 2022.
https://www.seattletimes.com/education-lab/in-first-assessment-since-the-pandemic-began-washington-student-test-scores-drop-significantly/

We can see from the results of the most recent test scores in Washington State that we have much work to do with all students, especially BIPOC (Black and Indigenous People of Color) students or whose families are considered low income. Inequities were already present pre-pandemic, and unfortunately the pandemic has exacerbated the challenges of students in all categories.



**Native American/Alaskan Native children saw a particularly steep drop in participation (11 percentage points) between 2019 and 2021.*

Chart: Lauren Flannery / The Seattle Times • Source: Washington Office of Superintendent of Public Instruction

Why are so many of these students needs’ not being met? We must work harder in our society to inspire all children to learn math, and be willing to reconsider the methods that our traditional education system uses do not work for so many of our students. Students need to connect math to their daily lives, and to experience math hands-on and in 3D. This typically requires in-person support that has been lacking throughout the pandemic.

I know, as a former teacher in our public schools, it can seem like there is little time for creative approaches like project based learning, and there are no available resources for math related field trips. Teachers are pressured by standardized testing to teach children to repeat specific procedures to get correct answers. Teaching for testing leaves little room to encourage students to use other means to demonstrate an understanding of mathematical concepts – methods such as discussing and arguing why one answer is correct and another is not quite right, or using math to create art and music or writing an essay to prove their mathematical knowledge. Standardized test scores are useful to compare and track achievement, but are they the best way to assess a child’s true learning? Or to motivate them to do better?

To engage students, embrace their cultural assets, and inspire them to learn about math, let's examine D'Ambrosio's call to action:

> Mathematics is a compilation of progressive discoveries and inventions from cultures around the world during the course of history. Its history and ethnography form a wonderful mosaic of cultural contributions. Today, we too are playing a part in the evolution of the discipline of mathematics. It is time for educators to improve their understanding of the role that culture has played and continues to play in shaping mathematical development. It is time for educators to empower their students with this vital knowledge.[3]

What can we do to solve this challenging math problem? We can provide more culturally responsive ways for students to approach and learn math. To do this we must start with the belief that all children can learn math, and that positive emotional experiences with math are a key to a student's motivation to learn.

At Seattle Universal Math Museum (SUMM) we aim to provide play-based math, demonstrating the fun and creative nature of the subject, while giving students a hands-on learning experience and access to the historical aspects of mathematics that relate to their ancestries. With this knowledge and experience, students will be more motivated to engage with math. Teachers and parents need to rise to the challenge and be open to a broader approach in teaching, learning and the appreciation of the diversity of the subject of mathematics.

For more information about SUMM, and how we plan to spark each and every person to love math, please visit our website at www.seattlemathmuseum.org.

[3] D'Ambrosio, Ubiratan. "What is ethnomathematics, and how can it help children in schools?" *Teaching Children Mathematics*; Reston; Feb 2001. https://www.researchgate.net/publication/284702127_What_is_ethnomathematics_and_how_can_it_help_children_in_schools

The Art of Destroying Flexagons

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Flexagons are (usually) flat models, often made of paper, that can be folded in certain ways to reveal additional faces besides the two that were originally the back and front of the model. Flexing is the word used to describe the special paper folding manipulation that has to be done in order to reveal a new face. Flexagons were invented and studied by Arthur Stone and some prominent fellow math graduate students from Princeton University in 1939, popularised by Martin Gardner in 1956[1] and again in 1988[2], and since studied by many recreational mathematicians (c.f. references in Wikipedia[3]). Once constructed, flexagons are beautiful objects, especially when their faces are colored, and flexing them is real fun and leads to some very interesting structures.

Surprisingly enough, even destroying flexagons can be fun *and* artistic! We suggest 4 different ways to artistically destroy flexagons, each with its own merit. Our exchange gift is a set of 4 flexagons strips, one for each demonstration.

1. Cutting through the center of flexagon leafs (recommended strip: tri-hexa-flexagon).

Fold a tri-hexa-flexagon. Then, starting from the outside hinge of a 2-leaf pat, make a cut through the center of the leaves, traversing all the leaves until you return to the initial cutting point. The result is a trefoil knotted, 8 times half-folded Möbius strip. This is because the tri-hexa-flexagon is analogous to a 3 half-twisted Möbius strip[4].

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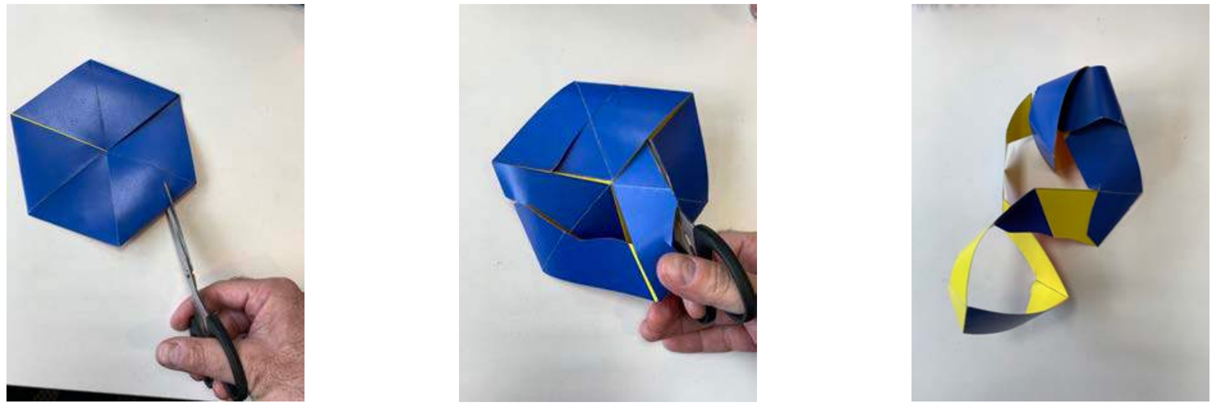
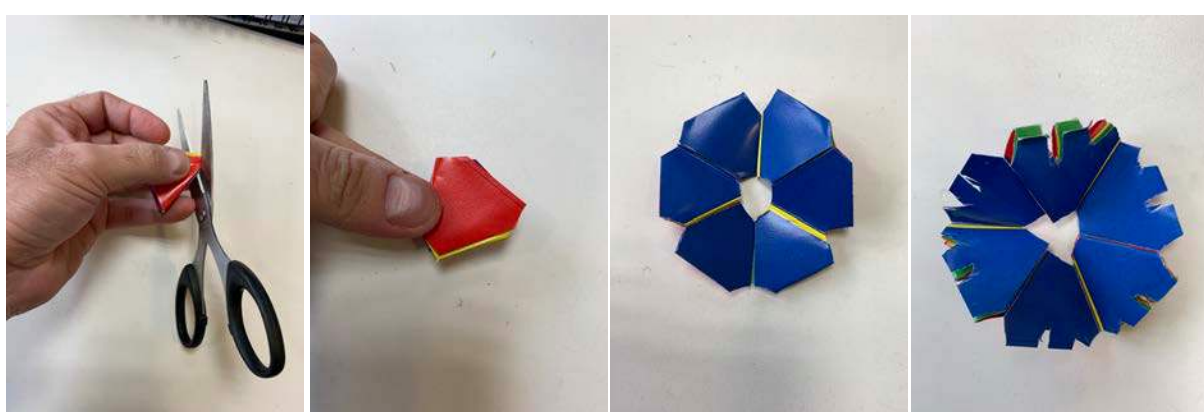


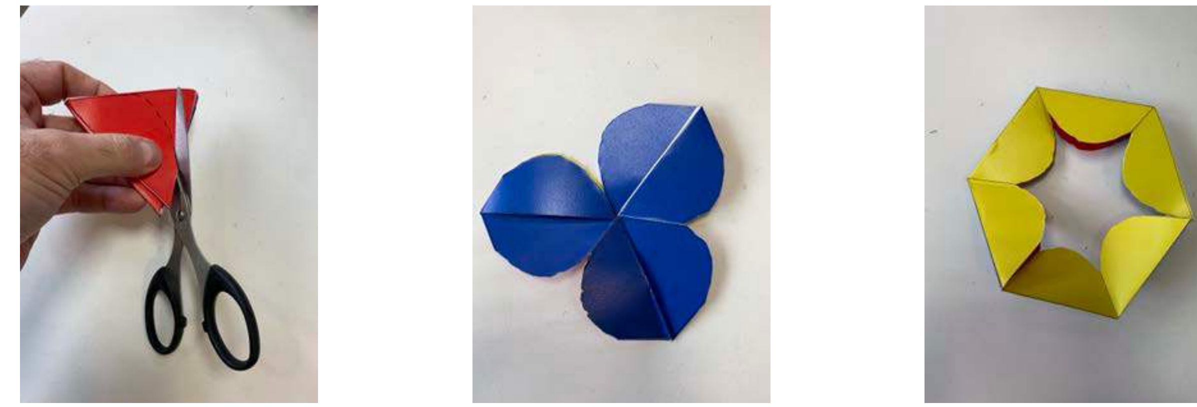
2. Cutting the corners and edges of flexagon leafs (recommended strip: hexa-hexa-flexagon).

Fold a hexa-hexa-flexagon. Then, stack all the leaves into one stack. The stack should be 18 leaves thick. Cut off each of the three corners of the flexagon. After cutting each corner it is recommended to open the flexagon and see the patterns that you get. We could call these "holy flexagons". Cutting off the corners of flexagons by just clipping the triangles' vertices is recommended as it eases flexing later on. The resulting flexagon can be further artistically enhanced by cutting patterns along the flexagons edges as well



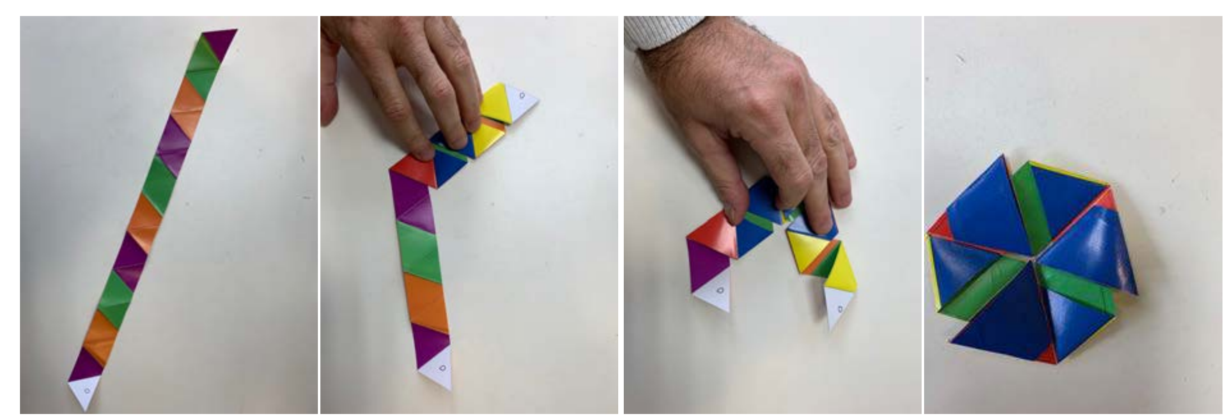
3. A circular cut in a flexagon (recommended strip: tri-hexa-flexagon).

Fold a tri-hexa-flexagon. Then, stack all the leaves into one stack. The stack should be 9 leaves thick. Make a circular cut, completely removing two of the flexagon's corners. Open up the result - it is a "flexa-frame"!



4. Cutting the flexagon before folding it (recommended strip: truncated hexa-hexa-flexagon).

Fold a hexa-hexa-flexagon from the truncated hexa-hexa-flexagon strip. Make sure that you still fold along the lines, even though (obviously) there will now be overlaps between different coloured leaves. The resulting flexagon can be flexed as usual, but the colourful patterns are very beautiful and provide insight into the inner structure of flexagons.



Can the Number of Pieces in a Rectangular Jigsaw Puzzle be a Multiple of the Number of Edge pieces?

By D. Scott Hewitt 2/2/2022

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The short answer is yes. There are an infinite number of solutions. However, some interesting results appear in connection to this question. I occasionally do Jigsaw puzzles, and I usually work on the edge pieces first. That's how I became interested in this topic.

Note: If you are new to jigsaws, there are quite a few options for doing Jigsaw puzzles. There are many available in stores or online. More choices are available online when you do an internet search for software. Some are free and some require the purchase of a license. I like the option of creating my own puzzles from jpg files, which some Jigsaw companies offer.

Let's start with some definitions and formulas:

Let x = the number of pieces on the shorter side.

let y = the number of pieces on the longer side.

let e = the number of edge pieces.

let f = the ratio of total pieces to edge pieces.

Let d = y – x (the difference between the long side and the short side).

Now let t = x y = total pieces.

e = 2x + 2y – 4 (true for any rectangular jigsaw puzzle)

f = $\frac{t}{e}$

We are interested in positive integer solutions for f.

We can assume that x > 1 because if x = 1 we just get a single row of pieces which is trivial.

Before talking about solutions, we have a few theorems in order to hopefully eliminate some possibilities: Most of the theorems are fairly basic and the proofs are short.

Theorem 1 is somewhat trivial and concerns jigsaw puzzles with only 2 rows of pieces.

Theorem 1 If and only if x = 2 and y is any positive integer, then f = 1 and t = e

Proof: t = 2y and e = 2*2 +2y -4 = 2y

So f = $\frac{t}{e} = \frac{2y}{2y}$ = 1, and every piece in the puzzle is an edge piece.

Note: If x > 2, then of course t > e.

Theorem 2 If x = 3, There is no solution for integer f.

Proof: t = 3y, and f = $\frac{t}{e} = \frac{3y}{6+2y-4} = \frac{3y}{2y+2}$

Since y ≥3, this fraction f takes on values on the interval $[\frac{9}{8}, \frac{3}{2})$ as y takes on values from 3 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 3.

Theorem 3 If x = 4, there is no solution for integer f.

Proof: t = 4y, and f = $\frac{t}{e} = \frac{4y}{2(4+y)-4} = \frac{4y}{2y+4} = \frac{2y}{y+2}$

Since y ≥ 4, this fraction f takes on values on the interval $[\frac{4}{3}$, 2) without reaching 2, as y takes on values from 4 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 4.

Theorem 4 x and y cannot both be odd.

Proof: If x and y are both odd then t is odd. We have f = $\frac{t}{2x+2y-4}$

The numerator is odd and the denominator is even, so x and y both odd is not possible.

Theorem 5 x and y cannot be equal, so square jigsaw puzzles are not possible with integer f.

Proof: $t = x^2$

$f = \frac{t}{e} = \frac{x^2}{4x-4}$

Case I

In the case of an even square where x = 2m, we have $\frac{4m^2}{8m-4} = \frac{m^2}{2m-1}$

If this fraction is equivalent to an integer, then $k(2m-1) = m^2$

We have: $m^2 - 2km + k = 0$ so $m = k + \sqrt{k(k-1)}$

For m to be an integer k(k-1) must be a perfect square.

However, consecutive integers do not share any factors. Therefore, k and k-1 must be consecutive perfect squares which is impossible.

Case II

In the case where x = 2m + 1, we have $f = \frac{4m^2+4m+1}{8m}$, but this is also impossible since an even number cannot divide an odd number. Therefore, x and y cannot be equal.

Theorem 6 y cannot be a multiple of x.

Proof: Suppose y = ax, then $t = xy = ax^2$

$e = 2(x+ax) - 4$ or $e = x(2+2a) - 4$

We now have $f = \frac{x*x*a}{x(2+2a)-4}$ and we know that $x \geq 5$ from previous work.

If x = 5 we have $f = \frac{25a}{5(2+2a)-4}$ and a number of the form 25r or 5r cannot be divisible by a number of the form 5r - 4.

If x = 6 we have $f = \frac{36a}{6(2+2a)-4}$ and again, a number of the form 36r or 6r cannot be divisible by a number of the form 6r – 4

For $x \geq 7$ let us look at the general case $f = \frac{x*x*a}{x(2+2a)-4}$ The numerator is of the form xr and the denominator is of the form xr – 4 . Numbers of the form xr – 4 can never divide numbers of the form xr with $x \geq 7$.

Therefore, for all $x \geq 5$, f cannot be an integer. This proves that y cannot be a multiple of x.

Now let us look at a condition that does yield solutions. Suppose x and y have a difference of 2.

Theorem 7 If d = y - x = 2 there are an infinite number of solutions.

Proof: If y - x = 2 then t = x(x+2) and e = 2x +2(x+2) – 4 = 4x

We have $f = \frac{x(x+2)}{4x} = \frac{x+2}{4}$ and we can solve for any f.

Example A: Suppose we wish the number of edge pieces to be exactly half the total number of pieces. So, f is 2 and we have $\frac{x+2}{4} = 2$

X+2 = 8

X = 6 and therefore y =8

t = 48 and e = 24

Example B: suppose we wish f to be 3.

We have $\frac{x+2}{4} = 3$

X + 2 = 12

X = 10 and therefore y = 12 giving us t = 120 and e = 40

Example C: suppose we wish f to be 73.

We have $\frac{x+2}{4} = 73$

X + 2 = 292

X = 290 and y =292 giving us t = 84680 and e = 1160

The following table gives a solution for f =2 through 20 where y - x = 2. Incidentally, these are also the jigsaws of smallest area with those f values. Other solutions exist for these f values when y – x > 2.

Table 1

f = t/e	x = width	y = length	t = total pieces	e = edge pieces
2	**6**	**8**	**48**	**24**
3	**10**	**12**	**120**	**40**
4	**14**	**16**	**224**	**56**
5	**18**	**20**	**360**	**72**
6	**22**	**24**	**528**	**88**
7	**26**	**28**	**728**	**104**
8	**30**	**32**	**960**	**120**
9	**34**	**36**	**1224**	**136**
10	**38**	**40**	**1520**	**152**
11	**42**	**44**	**1848**	**168**
12	**46**	**48**	**2208**	**184**
13	**50**	**52**	**2600**	**200**
14	**54**	**56**	**3024**	**216**
15	**58**	**60**	**3480**	**232**
16	**62**	**64**	**3968**	**248**
17	**66**	**68**	**4488**	**264**
18	**70**	**72**	**5040**	**280**
19	**74**	**76**	**5624**	**296**
20	**78**	**80**	**6240**	**312**

Table 2 gives the solutions for jigsaw puzzles of minimum area with consecutive x values where x = width. Notice that when x is a prime such as 31 or 43, or when x is an odd square like 49; y is sometimes quite large compared to neighboring y values due to the more difficult task of finding a solution with integer f.

Table 2

X = width	Y= length	t = x y	e = edge	f = t/e
5	12	60	30	2
6	8	48	24	2
7	30	210	70	3
8	18	144	48	3
9	14	126	42	3
10	12	120	40	3
11	24	264	66	4
12	20	240	60	4
13	132	1716	286	6
14	16	224	56	4
15	26	390	78	5
16	42	672	112	6
17	36	612	102	6
18	20	360	72	5
19	306	5814	646	9
20	27	540	90	6
21	38	798	114	7
22	24	528	88	6
23	48	1104	138	8
24	44	1056	132	8
25	92	2300	230	10
26	28	728	104	7
27	50	1350	150	9
28	65	1820	182	10
29	60	1740	174	10
30	32	960	120	8
31	870	26970	1798	15
32	50	1600	160	10
33	62	2046	186	11
34	36	1224	136	9
35	44	1540	154	10
36	68	2448	204	12
37	150	5550	370	15

38	40	1520	152	10
39	74	2886	222	13
40	57	2280	190	12
41	84	3444	246	14
42	44	1848	168	11
43	1722	74046	3526	21
44	90	3960	264	15
45	86	3870	258	15
46	48	2208	184	12
47	96	4512	282	16
48	92	4416	276	16
49	282	13818	658	21
50	52	2600	200	13

From table 1 and table 2, we see that every x value greater than or equal to 5 yields at least one solution, and every f value greater than or equal to 2 yields multiple solutions.

Tables 3 and 4 are just to show that perfect squares and cubes are possible for the number of edge pieces.

Table 3 lists very specific cases where e is a perfect square and x ≤ 100. Although there appears to be an infinite number of solutions, there are only 5 solutions for x below or equal to 100. If we were to go a bit further, there are 18 solutions for x below 500.

Table 3

x	y	t = x y	e	f = t/e
18	56	1008	144	7
50	152	7600	400	19
98	296	29008	784	37
98	1472	144256	3136	46
100	2352	235200	4900	48

Table 4 lists cases where e is a perfect cube and x is less than 1000.

Table 4

x	y	t = x y	e	f = t/e
54	56	3024	216	14
162	704	114048	1728	66
250	252	63000	1000	63
686	688	471968	2744	172

Triangular numbers are numbers of the form T(n) = $\frac{n(n+1)}{2}$. If we ask whether both x and y can be triangular numbers, the answer is yes; but solutions seem to be somewhat scarce. Table 5 gives the only 8 solutions I could find with the restriction that both x and y are less than 50,000. I was unable to find a formula to calculate these directly. It's interesting that both T88 and T168 show up twice.

Table 5

x	y	t=x*y	e=edge	f	Triangular Index for x	Triangular Index for y
78	171	13338	494	27	T12	T18
903	1378	1244334	4558	273	T42	T52
1770	24310	43028700	52156	825	T59	T220
2850	3916	11160600	13528	825	T75	T88
3916	5253	20570748	18334	1122	T88	T102
11325	14196	160769700	51038	3150	T150	T168
14196	17578	249537288	63544	3927	T168	T187
26106	31375	819075750	114958	7125	T228	T250

Now let us examine differences between y and x. Using a computer program, I did a search and found solutions for every difference up to d = 250 with the exception of d = 1, 3, 4, and 6.

Table 6 gives minimum solutions for all known consecutive values of d = y – x up to 25.

Table 6

d = y-x	x = short side	y = long side	t = total pieces	e = edge pieces	f = $\frac{t}{e}$
2	6	8	48	24	2
5	9	14	126	42	3
7	5	12	60	30	2
8	12	20	240	60	4
9	35	44	1540	154	10
10	8	18	144	48	3
11	15	26	390	78	5
12	30	42	1260	140	9
13	11	24	264	66	4
14	18	32	576	96	6
15	104	119	12376	442	28
16	14	30	420	84	5
17	21	38	798	114	7
18	32	50	1600	160	10

19	17	36	612	102	6
20	24	44	1056	132	8
21	209	230	48070	874	55
22	10	32	320	80	4
23	7	30	210	70	3
24	132	156	20592	572	36
25	23	48	1104	138	8

I call d values which do not yield a solution "difference outlaws".

Hypothesis(A): d = 1, 3, 4, and 6 are difference outlaws.

Hypothesis(B): d = 1, 3, 4, and 6 are the <u>only</u> outlaws. Therefore, all other differences are possible. Some differences yield x values that are quite large. For example, d = 225 yields the solution x = 25199, y = 25424, and f = 6328 with no smaller solution.

If someone would like to work on hypothesis A or B, I can get you started:

d = 1 → f = $\frac{x^2+x}{4x-2}$ → x^2 - 4xf +x + 2f = 0

d= 3 → f = $\frac{x^2+3x}{4x+2}$ → x^2 - 4xf +3x – 2f = 0

d = 4 → f = $\frac{x^2+4x}{4x+4}$ → x^2 - 4xf +4x – 4f = 0

d = 6 → f = $\frac{x^2+6x}{4x+8}$ → x^2 - 4xf +6x – 8f = 0

The fractions for f are equivalent to the Diophantine equations on the right that we would like to prove have no solution for x and f as positive integers. My feeling is that these 4 Diophantine equations probably have integer solutions but that the solutions involve negative integers. Incidentally, these equations represent hyperbolic curves.

Let's take a last look at one of the outlaws: d = 1. This looks like it may be easier than d = 3, 4, or 6.

<u>Theorem 8</u>

X and y cannot be consecutive integers. In other words, d≠ 1.

Proof: Suppose d = 1, then we have f = $\frac{x(x+1)}{4x-2}$

Suppose we treat this as a quadratic equation and solve it for x. We have

x^2 +(1-4f)x + 2f = 0

Using the Quadratic Formula: x = $\frac{(4f-1)\pm\sqrt{(1-4f)^2-8f}}{2}$

To have a solution, the discriminant must be a perfect square: $(1\text{-}4f)^2$ – 8f = n^2

We have 1 – 16f + 16f^2 = n^2 and n must be odd for x to be a positive integer.

The left side of the equation can be written as $16(f^2\text{-}f)$ +1 and this is equivalent to $8(2f^2 \text{-}2f)$ +1

This looks encouraging since all odd squares must be of the form 8n+1. In fact, all odd squares are of the form 8T + 1 where T is a triangular number of the form $\frac{r(r+1)}{2}$

The question now is: Can $2f^2$ -2f ever be a triangular number?

If so, we have $2f^2$ -2f = $\frac{r(r+1)}{2}$

$4f^2$ -4f – r(r+1) = 0 and we must again resort to the quadratic formula.

f = $\frac{(4)\pm\sqrt{4^2+16r(r-1)}}{8} = \frac{4\pm 4\sqrt{r(r+1)+1}}{8}$

Looking at this fraction, r(r+1) must be equivalent to 8T = 8 $\frac{v(v+1)}{2}$ = 4v(v+1), and we are missing a factor of 4. We have encountered a contradiction, so jigsaw puzzles with integer f are not possible if y – x = 1.

Feel free to send me an email if you have a comment about my paper or if you have made progress on Hypothesis A or B that you would like to share.

<u>**In conclusion, I would like to wish you an enjoyable time working on your next jigsaw puzzle, regardless of whether or not the number of pieces is a multiple of the number of edge pieces! Have fun!**</u>

A SLICE OF SPACE

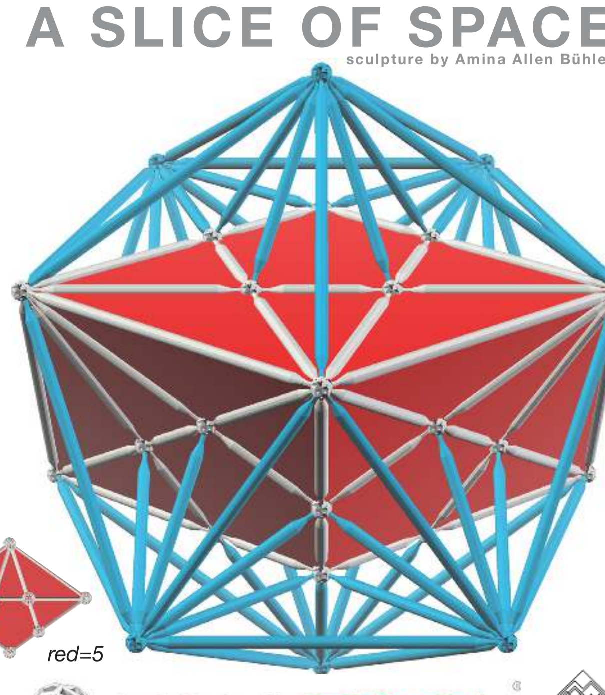
sculpture by Amina Allen Bühler

Dedicated to Steve Baer, Clark Richert, Marc Pelletier, John Conway and Roger Penrose

The *5 cubes of the dodecahedron* (a) bridge the apparent chasm between cubic (2-; 3-fold) and icosahedral (5-fold) symmetries. Amina Bühler-Allen recently discoved 10 “squashed” cubes* (c) hidden in one of Marc Pelletier's favorite models, the *stellated dodecahedron in an icosahedron* (b).

Each squashed cube has 6 rhombic faces, all “fat” Richert-Penrose tiles, and all lying in one of the Zometool's 6 red planes. Red planes are perpendicular to red lines, where 5-fold symmetry reigns.

*parallelepipeds



red=5

ZOMETOOL

In addition to building a model of Amina's sculpture, use this gift to build a number of quasi-periodic tilings that live in the red plane:

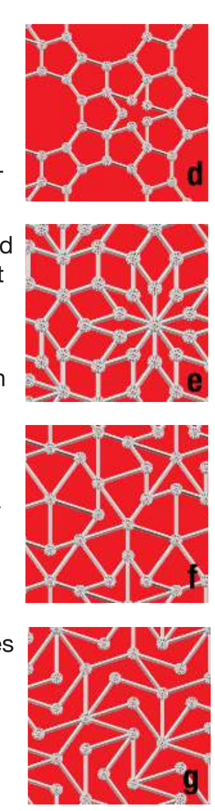
Johannes Kepler's tiling (d) of pentagons, stars and decagons,

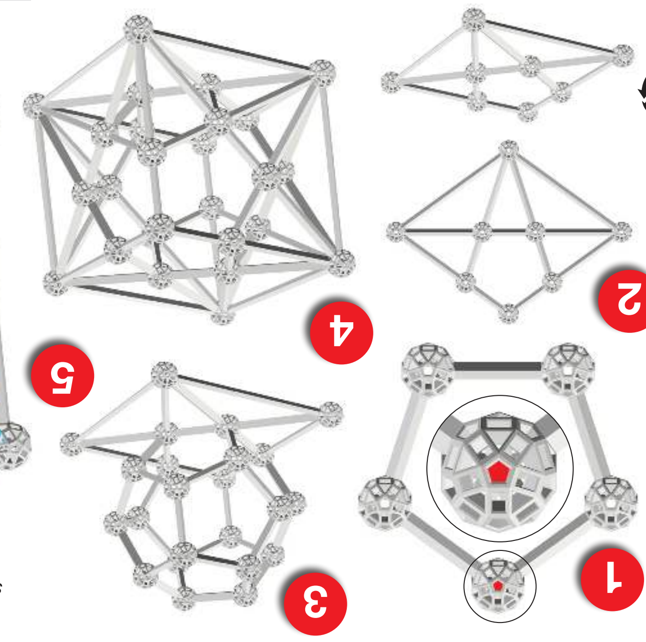
Richert-Penrose tiles (e), the fat and skinny rhombuses that Clark Richert discovered when casting a shadow of the rhombic “triacon” (try it!),

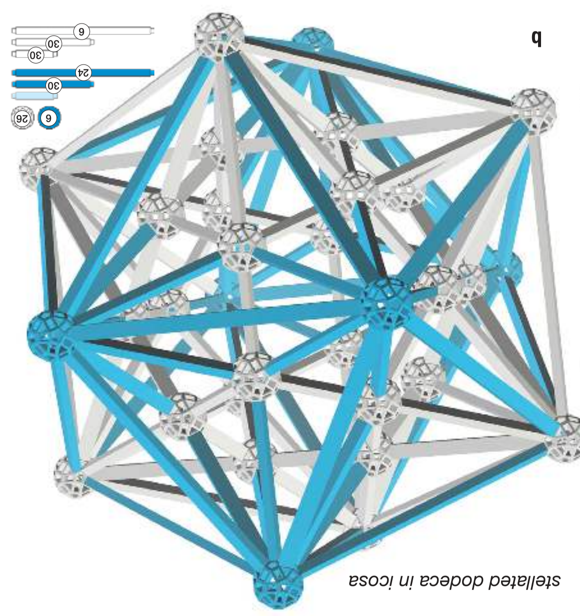
Penrose's kites and darts (f), which use 2 lengths of struts, and

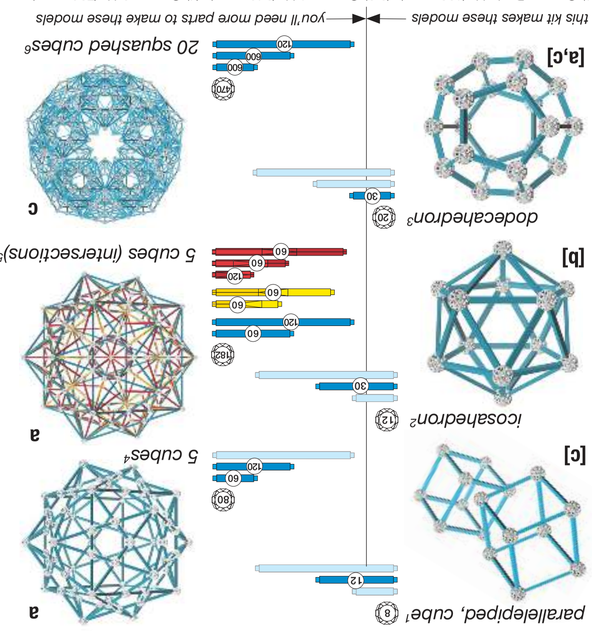
Clark Richert's “Star Pants” (g), a single (non-binary) tile that he hoped would tile the plane infinitely and quasi-periodically.

In another hyperplane, at a hyperangle to our own, Amina's cube isn't squashed at all. Since its edges are blue lines, it's projected from a 15-dimensional hypercubic lattice, where all 15 blue lines are perpendicular to each other.



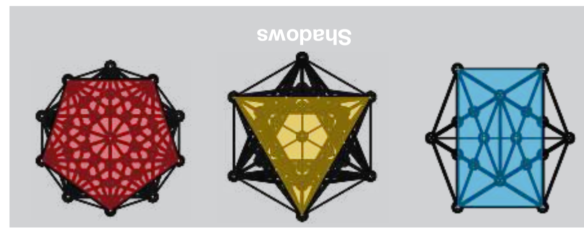


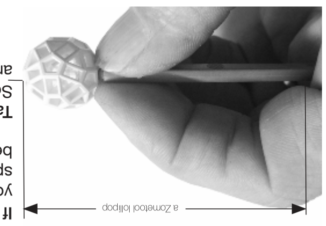




this kit makes these models → ← you'll need more parts to make these models

(1) Creator Basic kit (123 parts), (2,3) Creator 1 kit (246 parts), (4) Creator 3 kit (738 parts), (5) STEAM kit (1909 parts), (6) please email paulh@zometool.com for a quote





If it works, it works perfectly. Although you may bend a strut to insert it in a tight spot, finished models are “stress-free:” no bending, twisting or fudging necessary!

Take it apart, don't break it apart. Squeeze balls off struts with your thumb and forefinger.

Download free vZome software at vzome.com and never run out of parts. Heartfelt thanks to Scott Vorthmann and friends!

These images powered by vZome.

Even more thanks to Steve Rogers, David Booth, Mike Stranahan, Bob Nickerson, Dale Hess, Hildebrandt family, Donna Alpaugh, Jennifer Archuileta and Crystal B.

Generalization of Cone-pass and Continued Fraction
Cone-puter

February 2022

Akio Hizume

1. INTRODUTION

In my previous paper, "Cone-pass" [1], I presented a method for constructing the golden angle by manipulating a circular sheet with slits to create a cone. (Figure 1)

In addition to the "central angle *w*", which is a parameter that determines a cone shape, I brought in the concept of "overlap angle *w'* ". (Figure 2)

In order to consider both the central angle *w* and the overlap angle *w'* as real numbers in the $[0,1]$ interval, let's assume that the circumference is *1*, and thus we are dealing with a circular sheet of radius $r = 1/(2\pi)$. The circular sheet is an ideal paper with zero thickness, so no matter how many sheets are stacked on top of each other, there is no increase in thickness.

The central angle and the overlap angle generally do not coincide, as shown in Figure 2. However, if we continue recursively by finding the central angle from a cone with an appropriate overlap angle, creating a cone with the next overlap angle, and then finding a updated central angle, both the central angle and the overlap angle converge to the golden angle $(\tau\text{-}1) = (-1+\sqrt{5})/2 \approx 222.5°$. I named it the "recursive cone method" and showed in a previous paper that it corresponds exactly to the continued fraction expansion of the golden ratio. [1]

The cone with the golden angle as its central angle was called the "golden cone," and its elevation was composed of a right triangle with $r : r/\tau : \sqrt{(r/\tau)}$. (Figure 3, left)

As I noticed after the presentation of the previous paper, this triangle was the same as the so-called "Kepler's triangle" of $\tau : 1 : \sqrt{\tau}$. Kepler said, "There are two treasures in geometry. One is the Pythagorean theorem and the other is the golden ratio. The first may be compared to a gold nugget, and the second may be called a precious jewel." This triangle is truly a combination of gold and jewels.

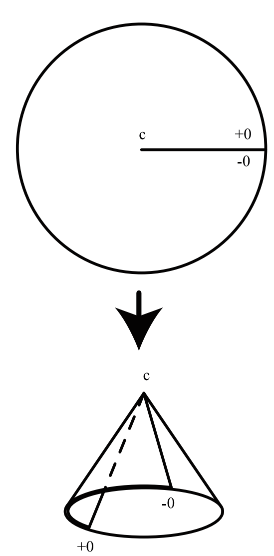


Fig.1
Make a cone from a circular sheet with a slit in it.

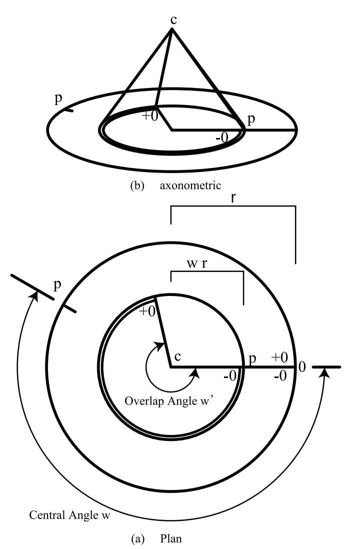
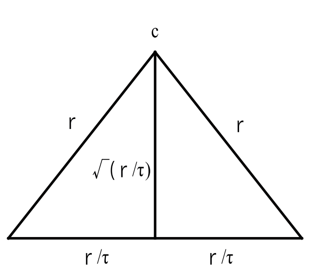


Fig. 2
Central Angle and Overlap Angle

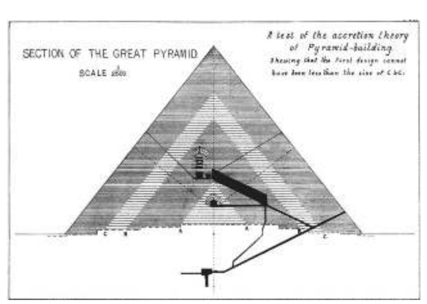
Kepler's triangle can also be seen in the elevation of the Pyramids of Giza (Fig. 3, middle). For a long time, I did not think this fact was very important. Since a right triangle with $\tau : 1 : \sqrt{\tau}$ can be easily constructed with a ruler and compass, it is no wonder that it appears everywhere in architecture. It's like if you draw a circle with a compass, you'll find the transcendental number π there.

However, as I myself wrote in my previous paper, "This triangle is fascinating and many other functions are likely to be discovered," I now believe that the use of Kepler's triangle in the pyramids was probably not an accident. It is a fact that this right triangle was constructed by the Egyptians thousands of years before Kepler, and the Egyptians of that time must have regarded this triangle as special. In fact, it may have had some function in the construction and mechanics of the pyramid, just as the geometric function of the golden angle was found to be constructed from Kepler's triangle.

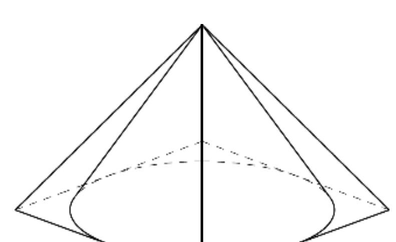
In this paper, I attempt to generalize the recursive cone method. The question is as follows.
What angles other than the golden angle can be determined by the recursive cone method?



The elevation of the golden cone is composed of Kepler's triangles



Petrie, W.M.F., The Pyramids and Temples of Gizeh, London 1883



The golden cone inscribed in the pyramid

Fig. 3

2. Generalization of the cone method

In the previous paper, we considered only the case where the paper overlaps twice when making a cone from a circular sheet with a slit in it. (Figure 2)

In other words, we did not consider the case where the central angle *w* of the cone is less than *1/2*. Let's lift this limitation and consider the problem.

Consider a circular sheet of circumference *1*, made of ideal paper of thickness *0*.
The radius of the circle is $r = 1/(2\pi)$. As shown in Fig. 4, the cut of the radius is set as the base point *0*, and the cut edges are set as *+0* and *-0*, respectively.
The circular paper is on a horizontal plane, which we call the surface plate, and an ideal axis of zero thickness is assumed to stand vertically at the center *c*.
The central angle *w* of the cone is defined to be *[0, 1]*. In other words, it can be any real number with $0 \leq w \leq 1$.
Choose an appropriate central angle w as shown in Fig. 4, and set the point of the circumference as P_1. Keeping the *-0* end of the slit aligned on the base point *c0*, slide the *+0* end clockwise to make a cone with P_1 and *-0* ends matched. (Figures 5 and 6)
Here, the reciprocal of the central angle *w*, *1/w*, is a value that should be called "overlap degree," meaning how much the papers overlap.
The overlap degree can be a real number with $1 \leq 1/w \leq \infty$.
The base radius of the cone made by the central angle *w* is *wr*.
When the central angle $w = 1$, i.e., the overlap degree is *1*, the cone is perfectly flat.
When the central angle *w* approaches zero infinitely, it becomes a cone wound infinite times and degenerates into a line segment of length *r*.

Let "*a* " be the integer part of *1/w*. *a* means the number of times that the *+0* end of the cone meets the *-0* end when it is slid to form a cone, so let's call it the "conjunction number *a*" (Fig. 5). The angle formed by the *(a+1)* overlapping pieces is called the overlap angle *w'* (Fig. 5).
The overlap angle *w'* is defined as follows

$$w' = (1-aw)/w \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Then, mark all the cone surfaces of *a* that match the edge of *-0* with $P_2, P_3, \ldots, P_a$ (Figure 5, Figure 6)
Open the cone and return it to the circular sheet. (Figure 7)
The markings made are lined up *a* times with central angles *w* from *-0* end..
It is obvious that the last remainder up to *+0* represents the fractional part of *1/w*.
The angles formed by the *a* marks and *-0* are, as follows in turn,

1st central angle $w(1)=w$
2nd central angle $w(2)=2w$
•
•
a th central angle $w(a)=aw$

Let's denote the *b*th central angle as $w(b)=bw$, where *b* is an integer with $1 \leq b \leq a$.

Let's call the remaining angle that is less than the central angle the "fraction angle." The value of a fractional angle is *1-aw*.
Note that Figures 4, 5, 6, and 7 are examples of $w=3/11$.
In Figures 5, 6, and 7, the cross section of the cone is drawn in a spiral shape for convenience, but as mentioned at the beginning, the cone is composed of ideal paper of zero thickness, so the thickness of the cone surface is zero no matter how many times it is wound.

We are now ready for consideration.

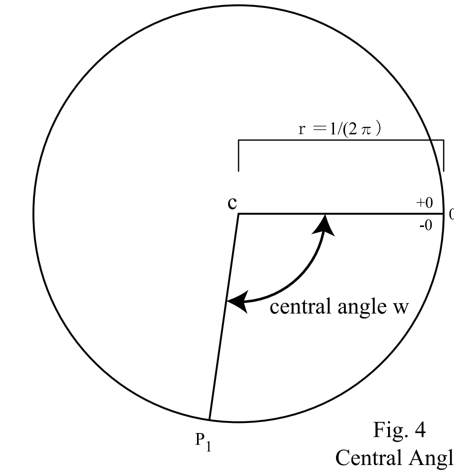


Fig. 4
Central Angle

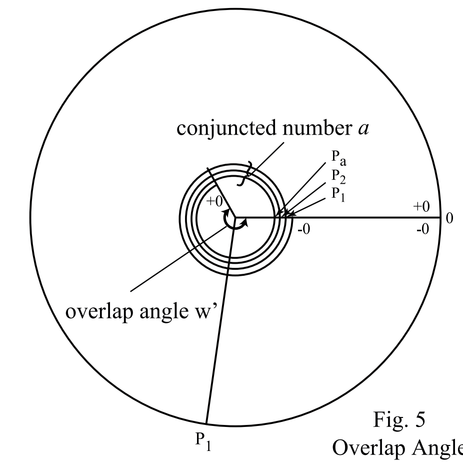


Fig. 5
Overlap Angle

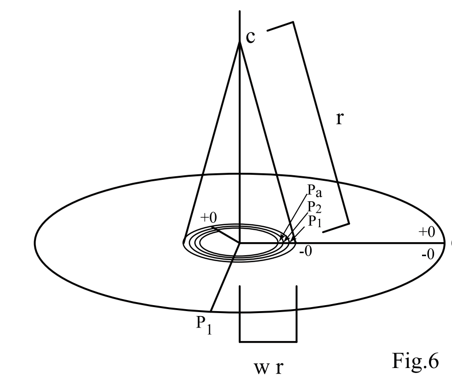


Fig.6
Axonometric of Cone

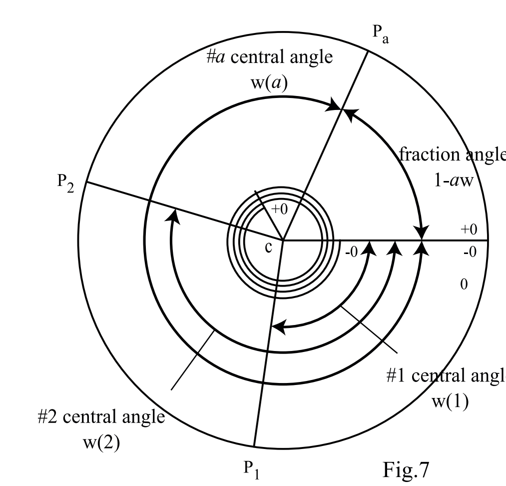


Fig.7

3. Construct an arbitrary regular *N*-gon / arbitrary rational angle

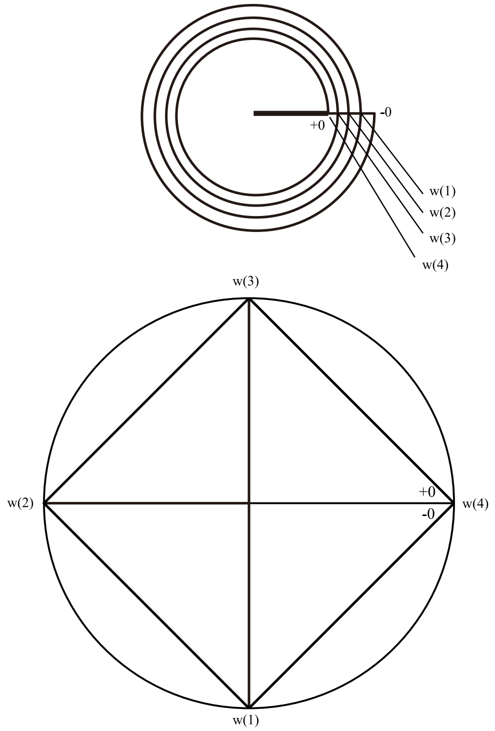


Fig. 8
Construction of a regular *N*-gon

Let us first consider the special case where the overlap degree $1/w$ is exactly integer N, as shown in Figure 8 above. The fraction angle $(1-aw) = 0$, and -0 and $+0$ make conjunction again. $a=N$ and the overlap angle $w' = 0$. As an example, Figure 8 on the left shows the case where $w(1)=1/4$ and $1/w=N=a=4$.

Mark all of the cone surfaces that match with the -0 end, return to the circular sheet, and connect the marked points on the circumference with lines in order to form a regular N-gon (Fig. 8, bottom).This is also self-evident. In other words, if you allow the cone method as a constructing method, you can construct any regular N-gon in a single operation.

Naturally, angles of rational numbers with N as the denominator can also be constructed. To find the angle of a rational number b/N, make a cone with exactly N overlap degree, and mark the bth central angle $w(b)$.

You can construct a regular pentagon with just a ruler and compass, but the next constructable regular polygons will have to wait until a regular *17*-gon. The fact was discovered by Gauss.

Later, it was known that "among regular N-gons where N is prime, such a construction is possible only if N is Fermat prime."

In addition, Gauss showed that "a necessary and sufficient condition for a regular N-gon to be constructable is in the form of a product of Fermat primes differing in power from N by *2*."

You can't even construct a regular *9*-gon with a ruler and compass.

Origami allows trisecting the corners, so you can fold a regular *9*-gon exactly. But even origami cannot fold arbitrary regular polygons.

4. Constructing irrational angles with the general recursive cone method: {*a*,*b*} operation

Next, let's consider the case where the overlap degree 1/w is not an integer.
$1/w > a$, resulting in an overlap angle. (Figure 9)
The equation to derive the central angle from the overlap angle is the inverse function of equation (1) .

$w=1/(a+w')$ ・・・・・・・・・・・・・(2)

This is the so-called *1*st central angle $w(1)$.
Therefore, after the second central angle are as folows.

2nd central angle $w(2)=2/(a+w')$
3rd central angle $w(3)=3/(a+w')$
・
・
ath central angle $w(a)=a/(a+w')$

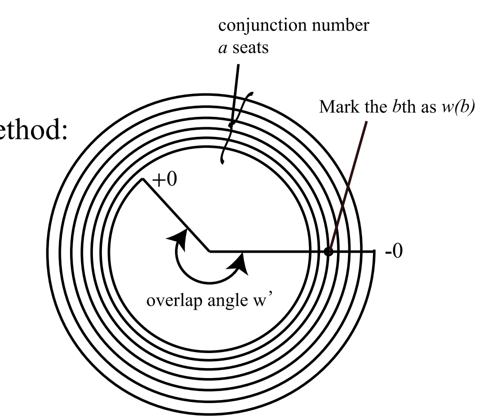


Fig. 9
General Recursive Cone Method
Schematic diagram of the {*a*,*b*} operation
Example of {*6*,*4*} operation

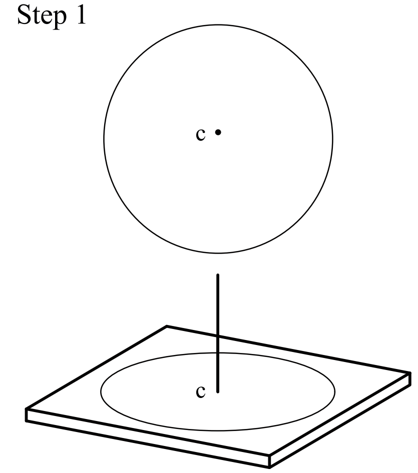
In general, the bth central angle is expressed as $w(b)=b/(a+w')$. (where b is an integer. $1 \le b \le a$)
Suppose that we now have a cone with an appropriate overlap angle w' and its conjunction number is a.(Figule 9)

1）Mark the bth cone surface matched with -0.
2）When opened in a circle, the angle between the marked point and -0 is the bth central angle. Mark the angle on the surface plate.
3）Raise the cone of conjunction number a again, and create a cone of overlap angle that match with the bth central angle. As before, make a new mark on the the bth cone surface matched with -0.
4）When opened in a circle, the angle between the marked point and -0 is the updated bth central angle. Mark the angle on the surface plate.
5）Return to the operation in 3) above, and repeat indefinitely thereafter.

In any initial form, if the above recursive operation with integer parameters a and b is repeated, the central angle and overlap angle will converge to the same angle. Let's call this recursive operation the "{*a*,*b*} operation".
In a previous paper [1] I showed a chart of the recursive cone method for constructing the golden angle. The next page shows the procedure (recipe) for the {*a*,*b*} operation, updated as a general recursive cone method.

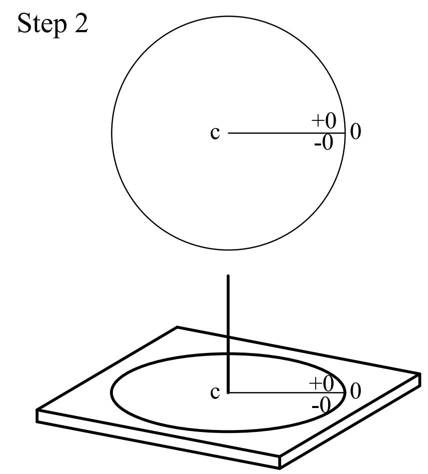
General Recursive Cone Method Procedure {*a*,*b*} operation

Step 1



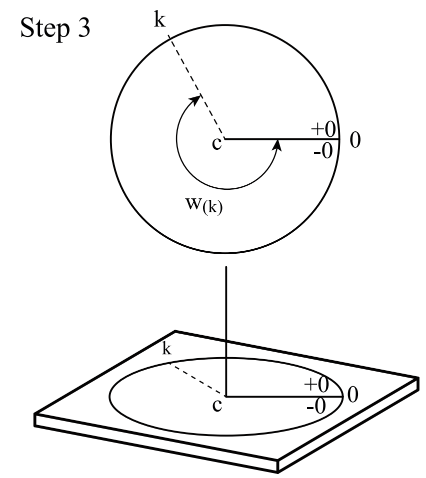
First, prepare a surface plate with a needle that stands vertically.
The thickness of the needle is ideally considered to be zero.
The position of the needle is c, and a circle of radius $r=1/(2\pi)$ is drawn on the surface plate with c as the center.
Since the height of the cone can never be longer than the radius of r, the length of the needle should be at least r.
The top figure is a plan view and the bottom figure is an axonometric.

Step 2



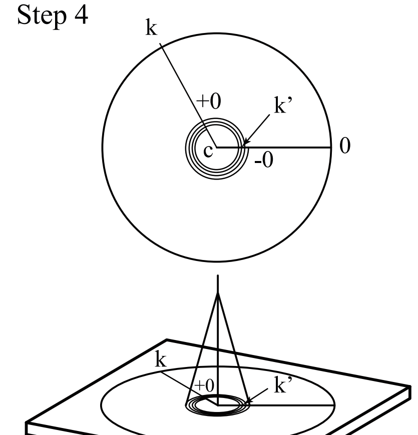
Cut out a circle of radius r from an ideal paper of thickness 0, and make a single slit from the center c toward the circumference.Thread the needle through the center of the paper circle and lay it on the surface plate.
Mark the base point 0 on the circumference of the surface plate located at the edge of the slit.
The edge point belonging to the upper part is $+0$, and the edge point belonging to the lower part is -0. The base point on the surface plate is assumed to be an unsigned 0.

Step 3



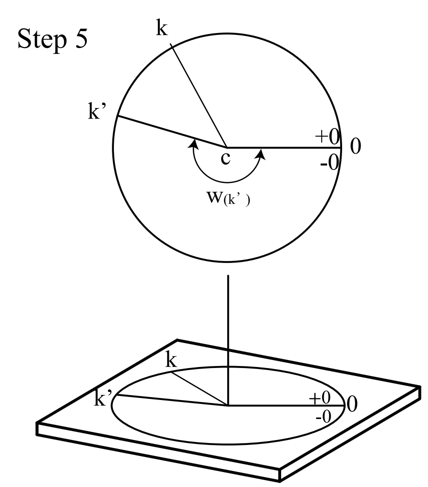
Mark "k" at an appropriate position on the circumference of the surface plate.
k is an integer. ($k \ge 1$)
The angle from the base point 0 to k in a clockwise direction is denoted as $w_{(k)}$. ($0 \leqq w_{(k)} \leqq 1$)

Step 4



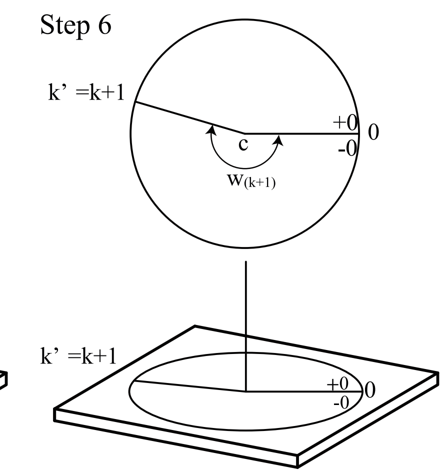
Draw a straight line ck from the center c of the circle on the surface plate to k.
Keeping the -0 end of the paper circle aligned on the base point $c0$, slide the $+0$ end clockwise and make a cone such that the $+0$ end lies on the line ck after winding a times.
This means that the overlap angle w' is set to the same angle as central angle $w(k)$.
Then mark k' on the bth cone surface matched with the -0 end.

Step 5



Return the cone to a flat surface again and align the slit with the base point 0.
The angle $w_{(k')}$ from starting point 0 clockwise to k' is as follows.
$w_{(k')}=b/(a+w_{(k)})$
This is the bth central angle of the cone.

Step 6



k' is set to $k+1$, then k in Step 3 is updated to the angle of $k+1$, and the same operation is repeated infinitely.

$w_{(\infty)}$ converges to $\frac{-a+\sqrt{a^2+4b}}{2}$

5. $\{a,b\}$ operations and continued fractions

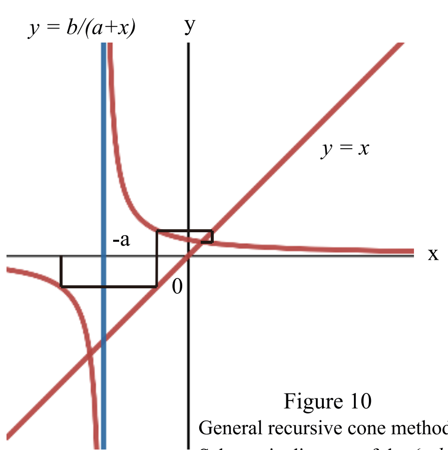


Figure 10
General recursive cone method.
Schematic diagram of the $\{a,b\}$

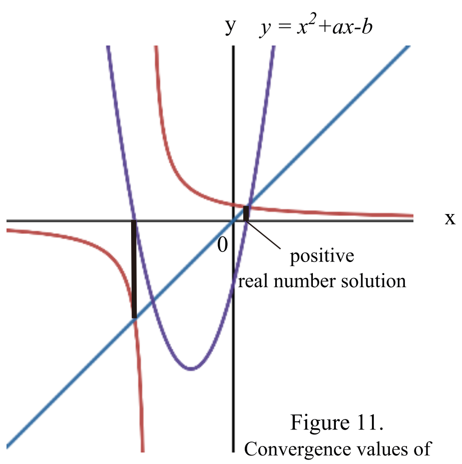


Figure 11.
Convergence values of
$\{a,b\}$ operations

The convergence of the $\{a,b\}$ operation with the recursive cone method can be understood from the hyperbola of $y=b/(x+a)$ and the line of $y=x$. The convergence can be visualized as a trajectory of recursive operations, where $y=b/(x+a)$ is obtained from the appropriate value of x, and then $y=b/(x+a)$ is updated with the new x.(Figure 10)
The value that converges is the positive real solution of $x^2+ax-b=0$. (Figure 11)
Therefore, if we repeat the $\{a,b\}$ operation infinitely, the convergence value is as follows.

$$w = w' = \frac{-a+\sqrt{a^2+4b}}{2} \quad \cdots\cdots\cdots\cdots (3)$$

The convergence angle list for the conjunction number a is 1, 2, 3,. .7 is shown in Table 1 on the next page.
In particular, when a is even, the central angle w = overlapping angle w' converged by the recursive cone method can be shown to cover the fractional part of the square root of a non-square natural number.
The value of $a/2$ is the integer part N of the square root of a natural number.

$$a=2N \quad \cdots\cdots\cdots\cdots (4)$$

(The case where the natural numbers are square numbers corresponds to the case where $1/w$ is exactly an integer in Section 3.)
Instead of relying on the list in Table 1, let's construct the angle corresponding to the fractional part of the square root of a natural number n that is not a square number.
First, we figure the integer part N of square root of the natural number n.
From equation (4), since $2N$ is the value of the conjunction number a, make a cone winding circular sheets as many times as a.
The value of b is as follows

$$b = n - N^2 \quad \cdots\cdots\cdots\cdots (5)$$

By using the general recursive cone method $\{a,b\}$ operation formulated in Section 4, we can construct the angle that is the fractional part of $\sqrt{n}$.
The recursive cone method $\{a,b\}$ operation is equivalent to recursively computing the following recurrence formula.

$$x_{k+1}=b/(a+x_k) \quad \cdots\cdots\cdots\cdots (6)$$

From the form of the recurrence formula (6), we can see that it can be expressed as a continued fractions with the partial denominator constant in a and the partial numerator constant in b. (Figure 12)
Following the reference [2], when the partial numerator of a continued fraction is a constant b, it is called a "b-continued fraction" and is denoted by $[C_1,C_2,C_3,\ldots]_b$. In particular, when the partial numerator is always 1, it is denoted by $[C_1,C_2,C_3,\ldots]_1$. This is the so-called "regular continued fraction".
The recursive cone method is a b-continued fraction, and in particular, the partial denominator is also a constant a, so it can be written minimally as $[\bar{a}]_b$. The upper line of $\bar{a}$ denotes that a repeats infinitely. The author thinks that this type of continuous fraction notation is more essential than regular continuous fractions. Only two integers are enough to informatize about an irrational number, and the convergence is fast. If I were a genetic designer, I would choose this over regular continued fractions.

The $\{a,b\}$ operation, or the continued fraction $[\bar{a}]_b$, can be used to calculate the convergents of the $\sqrt{n}$ you want to find, as desired. Let's find $\sqrt{83}$ as an example.
We can know by rote that the integer part N of $\sqrt{83}$ is 9. From equations (4) and (5), we get

$a = 2N = 2*9 = 18$
$b = n - N^2 = 83 - 9^2 = 2$

So if we calculate the continued fraction $[\overline{18}]_2$, we get the fractional part of $\sqrt{83}$, that is $-9+\sqrt{83}$.
The recurrence formula is as follows.

$x_{k+1}=2/(18+x_k)$

If we start with the initial value $x_0=0$,

$x_1=2/(18+0)=1/9$
$x_2=2/(18+1/9)=18/163$
$x_3=2/(18+18/163)=163/1476$

and a fairly good convergent can be obtained with only three recursive operations.
And similarly, the angle of $163/1476$ can be constructed exactly by the general recursive cone method $\{18,2\}$operation.

$$x_\infty = \cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cdots}}}}$$

Figure 12.
$\{a,b\}$ operation and equivalence of $[\bar{a}]_b$ continued fractions

Conjunction Number a	#b Central Angle b	The solution of $x^2+ax-b=0$ $\frac{-a+\sqrt{a^2+4b}}{2}$ Convergence Angle	Recurrence Formula $w'_{k+1}=\frac{b}{a+W'_k}$	$[\bar{a}]_b$ Continued Fraction $\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\cdots}}}$	Regular Continued Fraction $\cfrac{1}{C_1+\cfrac{1}{C_2+\cfrac{1}{C_3+\cdots}}}$	Approximate Value	Approximability
1	1	$\frac{-1+\sqrt{5}}{2}$	$w'_{k+1}=\frac{1}{1+W'_k}$	$[\bar{1}]_1$	$[\bar{1}]_1$	0.618033989	16
2	1	$-1+\sqrt{2}$	$w'_{k+1}=\frac{1}{2+W'_k}$	$[\bar{2}]_1$	$[\bar{2}]_1$	0.414213562	9
2	2	$-1+\sqrt{3}$	$w'_{k+1}=\frac{2}{2+W'_k}$	$[\bar{2}]_2$	$[\overline{1,2}]_1$	0.732050808	12
3	1	$\frac{-3+\sqrt{13}}{2}$	$w'_{k+1}=\frac{1}{3+W'_k}$	$[\bar{3}]_1$	$[\bar{3}]_1$	0.302775638	6
3	2	$\frac{-3+\sqrt{17}}{2}$	$w'_{k+1}=\frac{2}{3+W'_k}$	$[\bar{3}]_2$	$[\overline{1,1,3}]_1$	0.561552813	11
3	3	$\frac{-3+\sqrt{21}}{2}$	$w'_{k+1}=\frac{3}{3+W'_k}$	$[\bar{3}]_3$	$[\overline{1,3}]_1$	0.791287847	10
4	1	$-2+\sqrt{5}$	$w'_{k+1}=\frac{1}{4+W'_k}$	$[\bar{4}]_1$	$[\bar{4}]_1$	0.236067977	5
4	2	$-2+\sqrt{6}$	$w'_{k+1}=\frac{2}{4+W'_k}$	$[\bar{4}]_2$	$[\overline{2,4}]_1$	0.449489743	7
4	3	$-2+\sqrt{7}$	$w'_{k+1}=\frac{3}{4+W'_k}$	$[\bar{4}]_3$	$[\overline{1,1,1,4}]_1$	0.645751311	11
4	4	$-2+\sqrt{8}$	$w'_{k+1}=\frac{4}{4+W'_k}$	$[\bar{4}]_4$	$[\overline{1,4}]_1$	0.828427125	9
5	1	$\frac{-5+\sqrt{29}}{2}$	$w'_{k+1}=\frac{1}{5+W'_k}$	$[\bar{5}]_1$	$[\bar{5}]_1$	0.192582404	4
5	2	$\frac{-5+\sqrt{33}}{2}$	$w'_{k+1}=\frac{2}{5+W'_k}$	$[\bar{5}]_2$	$[\overline{2,1,2,5}]_1$	0.372281323	8
5	3	$\frac{-5+\sqrt{37}}{2}$	$w'_{k+1}=\frac{3}{5+W'_k}$	$[\bar{5}]_3$	$[\overline{1,1,5}]_1$	0.541381265	10
5	4	$\frac{-5+\sqrt{41}}{2}$	$w'_{k+1}=\frac{4}{5+W'_k}$	$[\bar{5}]_4$	$[\overline{1,2,2,1,5}]_1$	0.701562119	9
5	5	$\frac{-5+\sqrt{45}}{2}$	$w'_{k+1}=\frac{5}{5+W'_k}$	$[\bar{5}]_5$	$[\overline{1,5}]_1$	0.854101966	8
6	1	$-3+\sqrt{10}$	$w'_{k+1}=\frac{1}{6+W'_k}$	$[\bar{6}]_1$	$[\bar{6}]_1$	0.16227766	4
6	2	$-3+\sqrt{11}$	$w'_{k+1}=\frac{2}{6+W'_k}$	$[\bar{6}]_2$	$[\overline{3,6}]_1$	0.31662479	5
6	3	$-3+\sqrt{12}$	$w'_{k+1}=\frac{3}{6+W'_k}$	$[\bar{6}]_3$	$[\overline{2,6}]_1$	0.464101615	6
6	4	$-3+\sqrt{13}$	$w'_{k+1}=\frac{4}{6+W'_k}$	$[\bar{6}]_4$	$[\overline{1,1,1,1,6}]_1$	0.605551275	11
6	5	$-3+\sqrt{14}$	$w'_{k+1}=\frac{5}{6+W'_k}$	$[\bar{6}]_5$	$[\overline{1,2,1,6}]_1$	0.741657387	10
6	6	$-3+\sqrt{15}$	$w'_{k+1}=\frac{6}{6+W'_k}$	$[\bar{6}]_6$	$[\overline{1,6}]_1$	0.872983346	7
7	1	$\frac{-7+\sqrt{53}}{2}$	$w'_{k+1}=\frac{1}{7+W'_k}$	$[\bar{7}]_1$	$[\bar{7}]_1$	0.140054945	4
7	2	$\frac{-7+\sqrt{57}}{2}$	$w'_{k+1}=\frac{2}{7+W'_k}$	$[\bar{7}]_2$	$[\overline{3,1,1,1,3,7}]_1$	0.274917218	9
7	3	$\frac{-7+\sqrt{61}}{2}$	$w'_{k+1}=\frac{3}{7+W'_k}$	$[7]_3$	$[\overline{2,2,7}]_1$	0.405124838	6
7	4	$\frac{-7+\sqrt{65}}{2}$	$w'_{k+1}=\frac{4}{7+W'_k}$	$[\bar{7}]_4$	$[\overline{1,1,7}]_1$	0.531128874	8
7	5	$\frac{-7+\sqrt{69}}{2}$	$w'_{k+1}=\frac{5}{7+W'_k}$	$[\bar{7}]_5$	$[\overline{1,1,1,7}]_1$	0.653311931	10
7	6	$\frac{-7+\sqrt{73}}{2}$	$w'_{k+1}=\frac{6}{7+W'_k}$	$[\bar{7}]_6$	$[\overline{1,3,2,1,1,2,3,1,7}]_1$	0.772001873	9
7	7	$\frac{-7+\sqrt{77}}{2}$	$w'_{k+1}=\frac{7}{7+W'_k}$	$[\bar{7}]_7$	$[\overline{1,7}]_1$	0.887482194	7

Table 1
List of recursive cone $\{a,b\}$ operations

The "Approximability" in the rightmost column is a comparison with the difficulty of approximating the Golden angle (τ-1). It is the order of the convergent in the regular continued fraction expansion of each convergence angle, which is comparable to the approximation accuracy of the 16th convergent in the regular continued fraction expansion of the Golden angle.

6. $[\bar{a}]_b$ Continued Fraction and Regular Continued Fraction

The above recursive operations are easy to write as a computer program.
However, it is the focus of this paper to obtain an equivalent solution as the central angle of a cone rather than a computer. Cone -pass or should I call it Cone-puter? This idea did not come from the knowledge of continued fractions, but from the consideration of the geometric figure of a cone, which naturally led to the continued fraction expansion.
The simplicity of $[\bar{a}]_b$ contiued fraction is the simplicity of cone. Its convergence is more than that of the regular continuous fraction $[C_1,C_2,C_3,...]_1$.
Nevertheless, regular cotinued fractions, which are the mainstream in the study of continued fractions, have the great advantage of outputting the best convergents that are already irreducible, even though the procedure is somewhat more complicated.
I can't find a good way to convert a b-continued fraction with $b>1$ into a regular continued fraction. Wikipedia "Generalized Continued Fraction" shows how to convert the partial numerator to 1, but in that case, the partial denominator is not always an integer, so it is not a conversion to a regular continued fraction.
In the end, a good way to convert $[\bar{a}]_b$ continued fractions into regular continued fraction would be to start with an initial value of x_0 of 0 in the recurrence formula in (6), repeat the recursive operation several times to make convergents of enough high order, and then use Euclidean algorithm to make regular continued fractions again. There is no need to proceed with infinite Euclidean algorithm. If the value of the conjunctuion number a appears in the partial denominator, the regular continued fraction is fixed. The reason for this is that the sequence of partial denominators until the appearance of the conjunction number a repeats itself thereafter.
As a side note, the sequence of partial denominators that precede a is symmetrical (palindromic) [3]. This can be observed in Table 1. This phenomenon is also interesting and awaits geometrical clarification.
Let's illustrate the regular continued fractional transformation with the aforementioned $-9+\sqrt{83}$.
The first three of the convergents output in $[\overline{18}]_2$ are

The 1st convergent is $1/9$
The 2nd convergent is $18/163$
The 3rd convergent is $163/1476$

Using Euclidean algorithm to expand $163/1476$ into a regular continued fraction is $[9,18,9]_1$.
Therefore, the regular continued fraction of $-9+\sqrt{83}$ is $[\overline{9,18}]_1$. The first three convergents output from this regular continued fractions are as follows.

The 1st convergent is $1/9$
The 2nd convergent is $18/163$
The 3rd convergent is $163/1476$

This is consistent with that of $[\overline{18}]_2$.
However, the convergents output by a general $[\bar{a}]_b$ continued fraction are not always consistent with the convergents output by the regular continued fraction converted by the above method.
For example, the case of $\sqrt{7}$ is remarkable. As shown in Table 2, when comparing $[\bar{4}]_3$ and $[\overline{1,1,1,4}]_1$ in a continued fraction of $-2+\sqrt{7}$, none of the convergents match and convergence is much faster for the $[\bar{4}]_3$ continued fraction.

$-2+\sqrt{7}=0.645751311..$

$[\bar{4}]_3$ continued fraction	Value	Error	$[\overline{1,1,1,4}]_1$ Continued Fraction	Value	Error
3/4	0.75	0.1042486889	1/1	1	0.3542486889
12/19	0.631578947	−0.0141723637	1/2	0.5	−0.1457513111
57/88	0.647727273	0.0019759617	2/3	0.666666667	0.0209153556
264/409	0.645476773	−0.0002745384	9/14	0.642857143	−0.0028941682
299/463	0.645789474	0.0000381626	11/17	0.647058824	0.0013075125
463/717	0.645746007	−0.0000053045	20/31	0.64516129	−0.0005900207
494/765	0.645752048	0.0000007373	31/48	0.645833333	0.0000820223

Table 2
Comparison of $[\bar{a}]_b$ continued fraction and regular continued fraction $[\overline{C_1,C_2, ... ,a}]_1$ of $-2+\sqrt{7}$

To be able to use the cone method to construct the angle of a convergent calculated from a regular continued fraction. Suppose that its convergent q/p is represented by $[C_1, C_2, ... , C_k]_1$. The $\{a,b\}$ operation described above was assumed to be repeated infinitely, but in the case of a finite number of operations, let's denote it by $\{a , b\}_i$ with the number of times i. Then the convergent q/p, represented by a continued fractions in $[C_1, C_2, ... , Ck]_1$, can be constructed by the following sequence of $\{a , b\}_i$ operations.

$\{C_k, 1\}_1 \quad \{C_{k-1}, 1\}_1 \quad ... \quad \{C_2, 1\}_1 \quad \{C_1, 1\}_1$

For example, the fourth convergent $9/14$ in $-2+\sqrt{7}$ can be constructed using the sequence $\{4, 1\}_1 \quad \{1, 1\}_3$.
Moreover, even for any convergent of regular continuous fraction expansions without circulating partial denominators, such as transcendental numbers, the corresponding angles can be constructed by above cone method.
As noted in Section 3, the rational number q/p can be constructed directly from a cone with exactly p-fold overlap, but it is also valid to use regular continued fractions as above. This is because, although the number of operations is increased, the number of windings is reduced and the accuracy of the construction is practically improved.

7. Odd conjunction number a

Now let's consider the case where the integer part a of $1/w$, i.e. the conjunction number a is odd.
At the least conjunction number $a=1$, the central angle is only $w(1)=w$, and the angle of convergence of the $\{1,1\}$ operation, i.e. the limit of the $[\bar{1}]_1$ continued fraction, is none other than the recursive cone method of the golden angle discussed in the previous paper [1]. That is, the golden angle is the convergence angle of the simplest recursive cone method and the convergence value of the simplest $[\bar{a}]_b$ continued fraction.

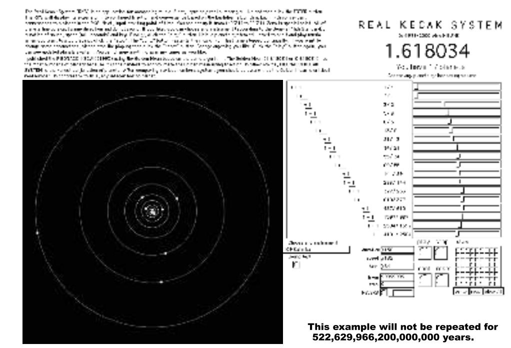


Fig. 14
Real Kecak System (2000)

If the conjunction number a shown in Table 1 is an odd number of 3 or more, the similar geometrical function as the golden angle is expected.
I've been having fun creating music that reflects its continued fractional structure. The application software Real Kecak System [4], developed in 2000, generates poly-rhythmic music that reflects the continuous fractional structure of any real number.
So far, apart from the golden ratio, I have explored the variety of quasi-periodic poly-rhythms by using the fractional part of the square root of integers, especially prime numbers, all of which have an even conjunction number a. Now I feel like I have discovered a new continent of poly-rhythmic music with odd conjunction numbers. The list in Table 1 seems to be a ranking of music in order of popularity. In first place, of course, is the music of the Golden Angle.

8. Constructing of line segments and list of Pisot numbers

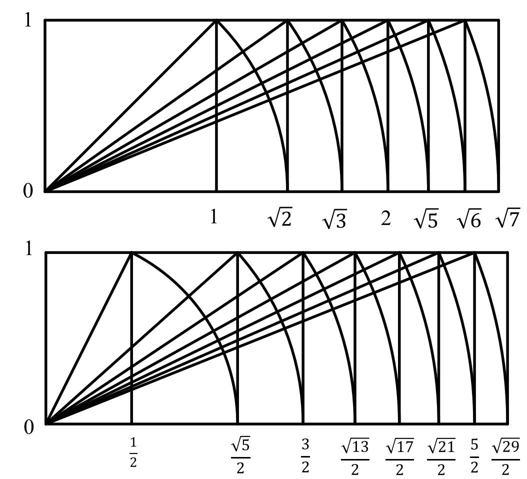


Figure 15.
Construction possible line lengths
with ruler and compass

The angle of the square root of a natural number n could be constructed by the recursive cone method for an even conjunction number a. It is known that the length of the square root of a natural number n can be constructed by a ruler and compass, as shown in Fig. 15 top. It can be done by starting from the diagonal of a square of side 1 and $n-1$ nested recursive constructions.
A similar nested constructions from the "diagonal of a $1 \times 1/2$ rectangle" gives the same values of the convergence angle as obtained by the recursive cone method with odd conjuction number a in Table 1.
The convergence values of even and odd conjunction numbers together coincide with the list of "quadratic irrational numbers which are Pisot numbers" [5].

9. Conclusion.

Can construct any regular polygon using the general recursive cone method and can construct any rational angle between 0 and 1.
The general recursive cone method can be used to construct an angle which is the fractional part of square root of any natural number. The desired angle can be obtained by recursively repeating the $\{a,b\}$ operation with two parameters: the even-conjunction number a and the bth central angle.
The general recursive cone method $\{a,b\}$ operation is equivalent to an $[\bar{a}]_b$ continued fraction with a constant partial denominator a and a constant partial numerator b.
The angle determined by the $\{a,b\}$ operation with odd-conjunction number a is expected to have similar geometric function as the golden angle.
The golden angle is the simplest angle that can be constructed by the general recursive cone method of odd conjunction numbers a.

Note

This paper is an intact machine translation of Hizume (2022, in Japanese)[6] . I did some proofreading, including unifying the terminology.

References

[1] As previous papers,
Akio hizume "Cone-pass," MANIFOLD #30 (2020).
Akio hizume "Cone-pass #2," ,MANIFOLD #31 (2020).
Akio hizume "Cone-pass," Bulletin of the Musashino Art University Vol. 51 (2021).
[2] Hiroaki Ito "Diophantine approximation by negative continued fraction," (2020).
[3] Ikuro's Homepage, In the article "The Problem of Continued Fraction Expansion (Part 2)," it is pointed out that in a regular continued fractions of the square root of any integer n, the integer part is N and the sequence of partial denominators is circular until $2N$ appears, and that the sequence before $2N$ is symmetrical.
This is the case in this paper where a is even. The same phenomenon is observed when a is an odd number.
[4] Akio hizume "Real Kecak System," MANIFOLD #2 (2001).
A ROM of the application software is included in the book "inter-native architecture OF music" (2006).
[5] Wikipedia "Pisot–Vijayaraghavan number."
[6] Akio Hizume "Generalization of Cone-pass and Continued Fraction" (in Japanese), MANIFOLD #33 (2022).

Open problem collection

Peter Kagey

January 16, 2020

This is a catalog of open problems that I began in late 2017 to keep tabs on different problems and ideas I had been thinking about.

Each problem consists of an introduction, a figure which illustrates an example, a question, and a list related questions. Some problems also have references which refer to other problems, to the OEIS, or to other web references.

1 Rating

Each problem is rated both in terms of how difficult and how interesting I think the problem is.

1.1 Difficulty

The difficulty score follows the convention of ski trail difficulty ratings.

	Easiest	The problem should be solvable with a modest amount of effort.
	Moderate	Significant progress should be possible with moderate effort.
	Difficult	Significant progress will be difficult or take substantial insight.
	Most difficult	The problem may be intractable, but special cases may be solvable.

1.2 Interest

The interest rating follows a four-point scale. Each roughly describes what quartile I think it belongs in with respect to my interest in it.

	Least interesting	These problems have an interesting idea, but may feel contrived.
	More interesting	Either a somewhat complicated or somewhat superficial question.
	Very interesting	Problems that are particularly natural or simple or cute.
	Most interesting	These are the problems that I care the most about.

For full set of problems:
https://www.gathering4gardner.org/g4g14gift/G4G14-PeterKagey-OpenishProblems.pdf

Problem 1.

Suppose you are given an $n \times m$ grid, and I then think of a rectangle with its corners on grid points. I then ask you to "black out" as many of the gridpoints as possible, in such a way that you can still guess my rectangle after I tell you all of the non-blacked out vertices that its corners lie on.

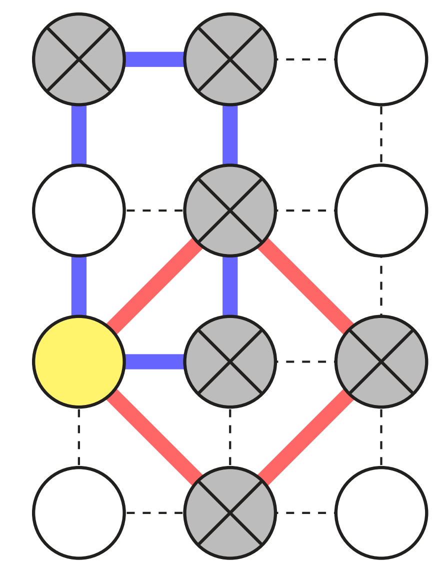


Figure 1: An example of an invalid "black out" for an 4×3 grid. The blue rectangle and the red rectangle have the same presentation, namely the gridpoint inside the yellow circle.

Question. How many vertices may be crossed out such that every rectangle can still be uniquely identified?

Related.

1. What if the interior of the rectangle is lit up instead?
2. What if all gridpoints that instersect the perimeter are lit up?
3. What if the rectangles must be square?
4. What if parallelograms are used instead of rectangles?
5. What if the rectangles must be horizontal, vertical, or 45° diagonal?
6. What if this is done on a triangular grid with equilateral triangles?
7. What if this is done in more dimensions (e.g. with a rectangular prism or tetrahedron?)

References.

`https://math.stackexchange.com/q/2465571/121988`

Problem 2.

Let G be some $n \times m$ grid as in Figure 1, where each cell has two opposite diagonals connected (uniformly at random). Choose a cell (also uniformly at random), and consider the component that goes through this cell.

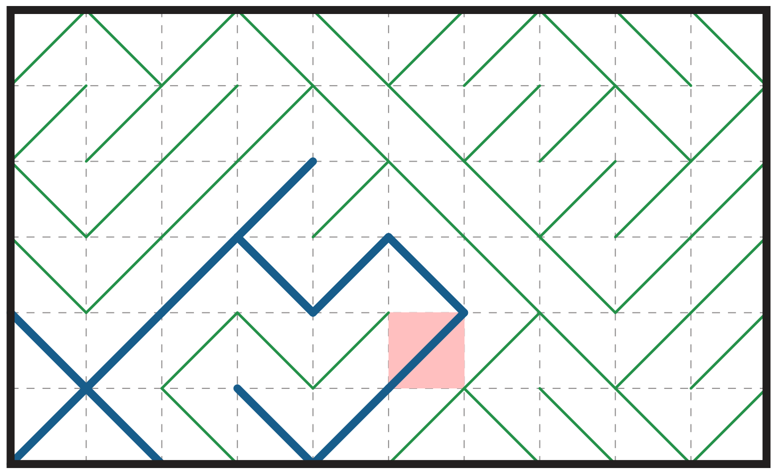


Figure 1: An example of a 6×10 grid, where a component of size 12 has been selected.

Question. What is the expected size of the selected component?

Related.

1. What is the expected number of components in an $n \times m$ grid?
2. How long is the longest component expected to be?
3. How does this change if the grid on a torus/cylinder/Möbius strip/etc?

Problem 3.

Peter Winkler's Coins-in-a-Row game works as following:

> On a table is a row of fifty coins, of various denominations. Alice picks a coin from one of the ends and puts it in her pocket; then Bob chooses a coin from one of the (remaining) ends, and the alternation continues until Bob pockets the last coin.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed according to some probability distribution.

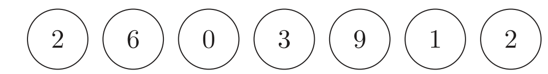


Figure 1: An instance of a seven coin game on a uniform distribution of $\{0, 1, \ldots, 9\}$. The first player has a strategy that allows her to win by one point.

Question. For some fixed ω, what is the expected first player's score of Peter Winkler's Coins-in-a-Row game when played with $X_1(\omega), X_2(\omega), \ldots, X_3(\omega)$ where both players are using a min-max strategy?

Note. Let

$$e = E[X_2 + X_4 + \ldots + X_{2n}] \text{ and } o = E[X_1 + X_2 + \ldots + X_{2n-1}]$$

When played with $2n$ coins, the first player's score is bounded below by $\max(e, o) - \min(e, o)$ by the strategy outlined by Peter Winkler.

Trivially the first player's score is bounded above by the expected value of the n largest coins minus the expected value of the n smallest coins.

Related.

1. If all possible n-coin games are played with coins marked 0 and 1, how many games exist where both players have a strategy to tie.
2. How does this change when played according to the (fair) Thue-Morse sequence?
3. What if the players are cooperating to help the first player make as much as possible (with perfect logic)?
4. What is both players are using the greedy algorithm?
5. What if one player uses the greedy algorithm and the other uses min-max? (i.e. What is the expected value of the score improvement when using the min-max strategy?)
6. What if one player selects a coin uniformly at random, and the other player uses one of the above strategies?

Problem 4.

Let a "popsicle stick weave" be a configuration of lines segments, called "sticks", such that

(1) when you lift up any stick by the end, the structure supports itself (is in tension)

(2) the removal of any stick results in a configuration that no longer supports itself.

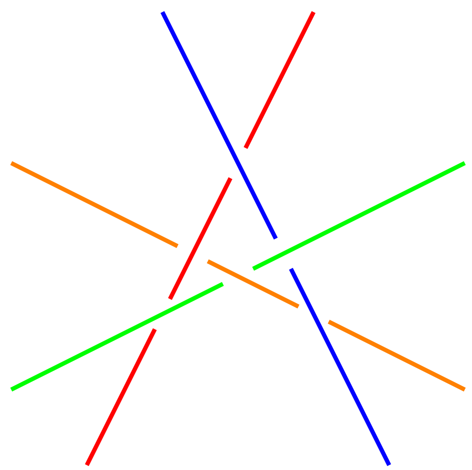


Figure 1: The unique example of a 4 stick crossing (up to reflection)

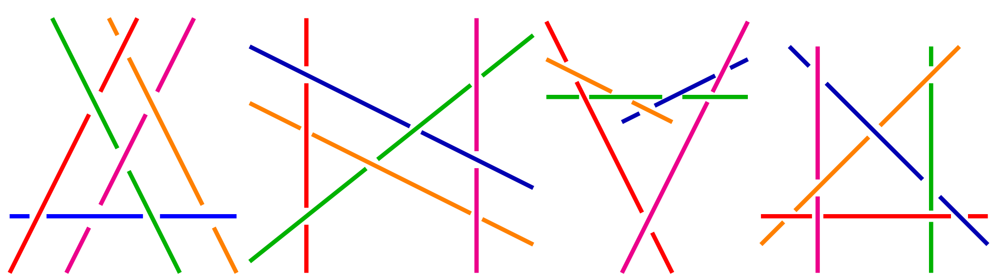


Figure 2: Four of five (?) known examples of five-stick crossings. Perhaps the fourth example shouldn't count, because shortening the blue stick to avoid the blue-red crossing results in a valid configuration (the remaining known five-stick crossing).

Question. How many distinct popsicle stick weaves exist for n sticks?

Related.

1. What if the sticks are only allowed to touch three other sticks?
2. What if the sticks are another geometric object (e.g. semicircles)?

Problem 5.

Let

$$C_n = \{f : [n] \to \mathbb{N} \mid \text{the convex hull around } \{(1, f(1)), \ldots, (n, f(n))\} \text{ forms an } n\text{-gon}\}$$

and then let $a(n)$ denote the least upper bound over all functions in C_n

$$a(n) = \min\{\max\{f(k) \mid k \in [n]\} \mid f \in C_n\}$$

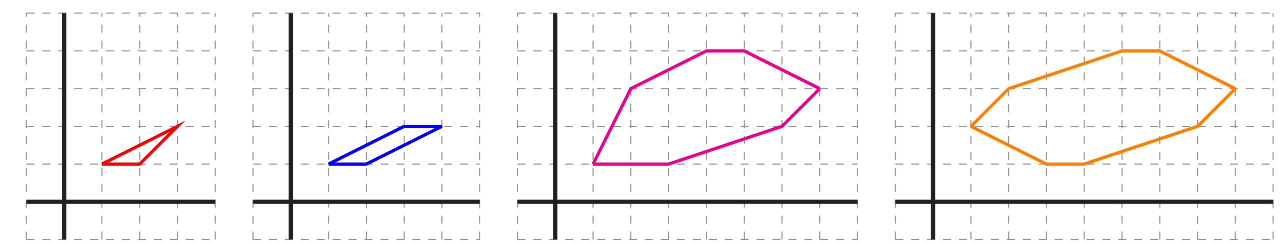


Figure 1: Examples of $a(3) = 2$, $a(4) = 2$, $a(7) = 4$, and $a(8) = 4$, where the polygons with an even number of vertices have rotational symmetry.

Question. Do these polygons converge to something asymptotically?

Related.

1. Does $a(2n) = a(2n - 1)$ for all n?
2. Do the minimal $2n$-gons always have a representative with rotational symmetry?
3. Are minimal $2n$-gons unique (up to vertical symmetry) with finitely many counterexamples?
4. What is the asymptotic growth of a(n)?

References.

A285521: "Table read by rows: the n-th row gives the lexicographically earliest sequence of length n such that the convex hull of (1, a(1)), ..., (n, a(n)) is an n-gon with minimum height." (https://oeis.org/A285521)

Problem 6.

Let $f_{n,m} \colon [n] \to [m]$ be a uniformly random function. Consider the convex hull around $\{(1, f(1)), \ldots (n, f(n))\}$

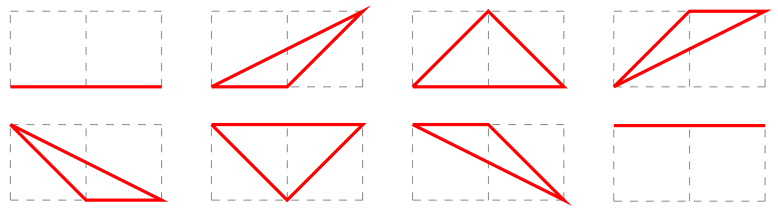


Figure 1: Examples of $f_{3,2}$. Here the expected number of sides on a convex hull is 2.75

Question. What is the probability of seeing a k-gon (for some fixed k), when given a uniformly random function $f_{n,m}$?

Related.

1. What value of k has the highest probability?
2. What is the expected value of the number of sides?
3. What if $f_{n,n}$ is restricted to be a permutation?
4. What if $f_{n,m}$ is injective?

Problem 7.

Given an $n \times n$ grid, consider all convex polygons with grid points as vertices. Let $m(n)$ be the greatest integer k such that there exists a convex k-gon on the $n \times n$ grid.

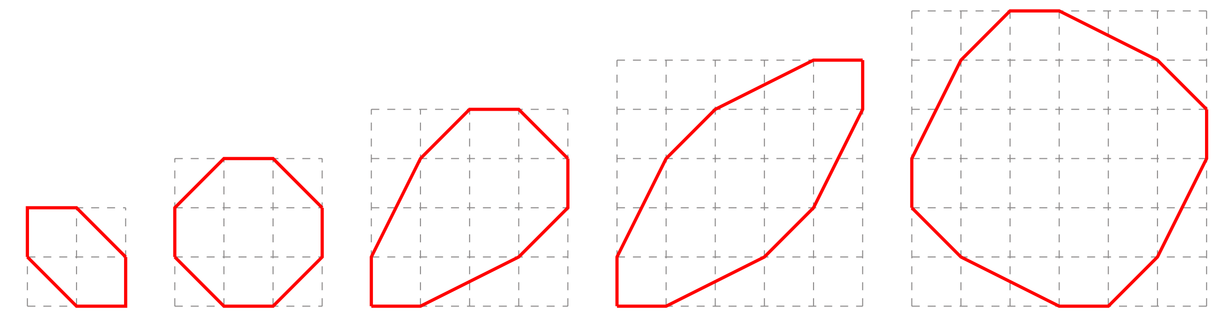


Figure 1: Examples that prove $m(3) = 6, m(4) = 8, m(5) \geq 9, m(6) \geq 10$, and $m(7) \geq (12)$

Question. What is $m(n)$?

Related.

1. What is a proof (or counterexample) that the examples shown are the best possible?
2. How does $m(n)$ grow asymptotically?
3. Do the shapes do anything interesting in the limit?
4. Are there finitely many maximal polygons without rotational symmetry (e.g. $m(5)$)?
5. How does this generalize to $m \times n$ grids?

References.

Problem 5.

Problem 6.

Problem 8.

Given an $n \times n$ grid, consider all the ways that convex polygons with grid points as vertices can be nested.

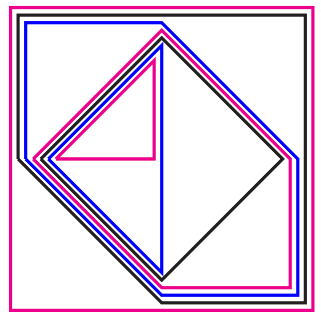


Figure 1: Seven nested convex polygons in the 3×3 grid.

Question. If we think of each polygon having the same height, what is the greatest volume that we can make by stacking the polygons this way?

Related.

1. What is the largest sum of the perimeters? The least?
2. What is the largest sum of the number of vertices? The least?
3. How many ways are there to stack $n^2 - 2$ polygons like this? Any number of polygons?
4. Does this generalize to polyhedra in the $n \times n \times n$ cube?
5. Does this generalize to polygons on a triangular grid?

Problem 9.

Consider all k-colorings of an $n \times n$ grid, where each row and column has $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ cells with each color.

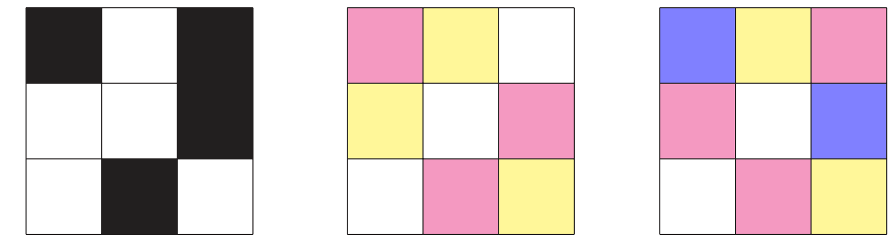


Figure 1: A valid 2-coloring, 3-coloring, and 4-coloring of an 3×3 grid.

Question. How many such k-colorings of the $n \times n$ grid?

Related.

1. What if there also must be a total of $\lfloor n^2/k \rfloor$ or $\lceil n^2/k \rceil$ cells of each color?
2. What if these are counted up to the dihedral action on the square D_4?
3. What if these are counted up to torus action?
4. What if these are counted up to permutation of the coloring?
5. Can this generalize to the cube? To a triangular tiling?

Problem 10.

Consider Ron Graham's sequence for LCM, that is, look at sequences such that

$$n = b_1 < b_2 < \ldots < b_t = k \text{ and } \mathrm{LCM}(b_1, \ldots, b_t) \text{ is square.}$$

Question. Let $A300516(n)$ be the least k (as a function of n) such that such a sequence exists?

$a(1) = 1$ via (1)	$a(11) = 121$ via $(11, 121)$	$a(21) = 49$ via $(21, 36, 49)$
$a(2) = 4$ via $(2, 4)$	$a(12) = 18$ via $(12, 18)$	$a(22) = 121$ via $(22, 64, 121)$
$a(3) = 3$ via $(3, 9)$	$a(13) = 169$ via $(13, 169)$	$a(23) = 529$ via $(23, 529)$
$a(4) = 4$ via (4)	$a(14) = 49$ via $(14, 16, 49)$	$a(24) = 48$ via $(24, 36, 48)$
$a(5) = 25$ via $(5, 25)$	$a(15) = 25$ via $(15, 16, 18, 25)$	$a(25) = 25$ via (25)
$a(6) = 12$ via $(6, 9, 12)$	$a(16) = 16$ via (16)	$a(26) = 169$ via $(26, 64, 169)$
$a(7) = 49$ via $(7, 49)$	$a(17) = 289$ via $(17, 289)$	$a(27) = 81$ via $(27, 81)$
$a(8) = 16$ via $(8, 16)$	$a(18) = 25$ via $(18, 20, 25)$	$a(28) = 49$ via $(28, 49)$
$a(9) = 9$ via (9)	$a(19) = 361$ via $(19, 361)$	$a(29) = 841$ via $(29, 841)$
$a(10) = 25$ via $(10, 16, 25)$	$a(20) = 25$ via $(20, 25)$	$a(30) = 50$ via (30)

Figure 1: Examples of $A300516(n)$ for $1 \le n \le 30$.

Related.

1. For what values n is $A300516(n)$ nonsquare?
2. For what values n does the corresponding sequence have three or more terms?
3. What is the analogous sequence for perfect cubes, etc?

References.

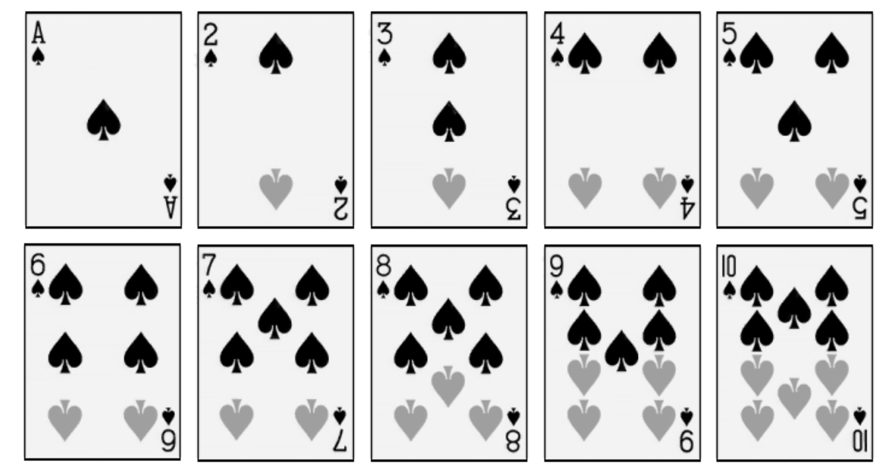
https://oeis.org/A300516

300+ Digits of π From an (Almost) Ordinary Deck of Cards

Mike Keith Jan 2022

In this paper we discuss a new number puzzle involving a standard deck of cards, one that turns out to be sufficiently difficult that it is initially not clear whether the desired construction is even achievable.

Preliminaries. In this puzzle we're going to use all 52 cards in a standard deck, which is comprised of 40 number cards (A through 10 of each suit) and 12 face cards (J, Q, K of each suit). The number cards have some special features that are important to our puzzle, so the A through 10 of spades are illustrated below.



The number that appears in two corners of each card (paired with a small suit symbol) is known as the *corner index*, and we'll call this number (1 (= A) through 10) its *value*, denoted by *v*. The suit symbols in the middle of each card are the *pips*. Note that some pips are rightside up and some are upside down; in the graphic above the upside-down pips are illustratively colored gray. The orientation of the pips shown here is the de facto standard for a deck of cards.

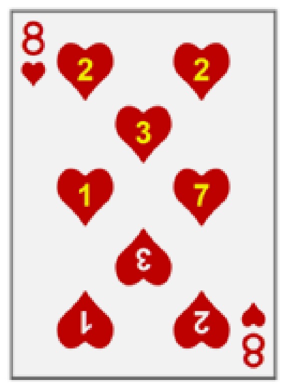
We can summarize these pip orientations by listing the *split* for each possible value, a pair of numbers (*r, u*) that specifies how many rightside-up pips (*r*) and upside-down pips (*u*) there are, where $r \geq v$ and $r + u = v$. The split numbers for $v = 1$ to 10 are shown in the table below. The cards shown above are oriented with the larger set of pips, corresponding to *r*, at the top.

Value (*v*)	1	2	3	4	5	6	7	8	9	10
Split (*r*,*u*)	1,0	1,1	2,1	2,2	3,2	4,2	5,2	5,3	5,4	5,5

Note that the 2, 4, and 10 cards, and only those, have $r = v$, so these cards look exactly the same when rotated by 180 degrees, but all the other cards are rotationally non-invariant. There are two distinct ways to place an A, 3, 5, 6, 7, 8, or 9 on a table: with the *r* pips facing up, or rotated by 180 degrees with the *u* pips facing up.

The numbered cards in a single suit have 1 + 2 + ... + 10 = 55 total pips, so the number of pips in all four suits is 55 x 4 = 220.

We now introduce the idea of *labeling* the pips. Imagine a single decimal digit (any digit 0 to 9) written inside each pip, with the orientation of each digit matching the orientation of the pip, so that if the pip is rightside up then so is the digit. Pips of the "up" and "down" orientation will be labeled with two different colors. We use the colors white and yellow, since these are both nicely visible when written inside either black (spades and clubs) or red (hearts and diamonds) pips. Here is an example of a pip-labeled card:



For the purposes of our puzzle, we're going to "read" the rightside-up part of each pip-labeled card as a sequence of decimal digits, by reading the corner index number first followed by the digits inscribed on the rightside-up pips in raster-scan order. The "10" index number on a ten card is read as the decimal digit "0", and an "A" is read as the digit 1. So the card above is read as "8 2 2 3 1 7", the 8 coming from the index number in the corner and the 22317 from the five yellow-numbered pips. Rotated 180 degrees this card becomes 8 2 1 3, from the 8 in the corner and the 213 on the white-numbered pips.

Note that, when numbering an asymmetric card (not a 2, 4, or 10), you can choose which "half" (*r* side or *u* side) gets the yellow numbers. The example above has the yellow numbers on the *r* side, but either way is acceptable when choosing how to number the pips.

If all 40 number cards are placed on a table in some order, face up, with all the pip labels of the same color on top, we refer to this as a *deal*. By definition the yellow numbers are on top in the first deal and the white numbers are on top in the second. Reading off the rightside-up digits (index number + pip numbers) of all the cards in the *k*th deal produces a sequence of n_k digits, for $k = 1$ and 2. Note that n_1 and n_2 need not be equal, but there are 220 pips and 80 corner indices, each of which contributes a digit, so the number of digits in both deals ($n_1 + n_2$) is 300.

Face cards. There is no straightforward way to interpret the indices of the face cards as decimal digits – especially since there are only three distinct indices – so we will ignore the indices on face cards but do a different kind of pip numbering, by placing zero to four digits in each of the two large pips which traditionally appear at upper left and lower right of the face card picture area. We picked four as a somewhat arbitrary upper limit by judging that it seems reasonable to put up to four digits in each large pip, but any more than four starts to look too crowded.

Here is an example of a pip-numbered face card:

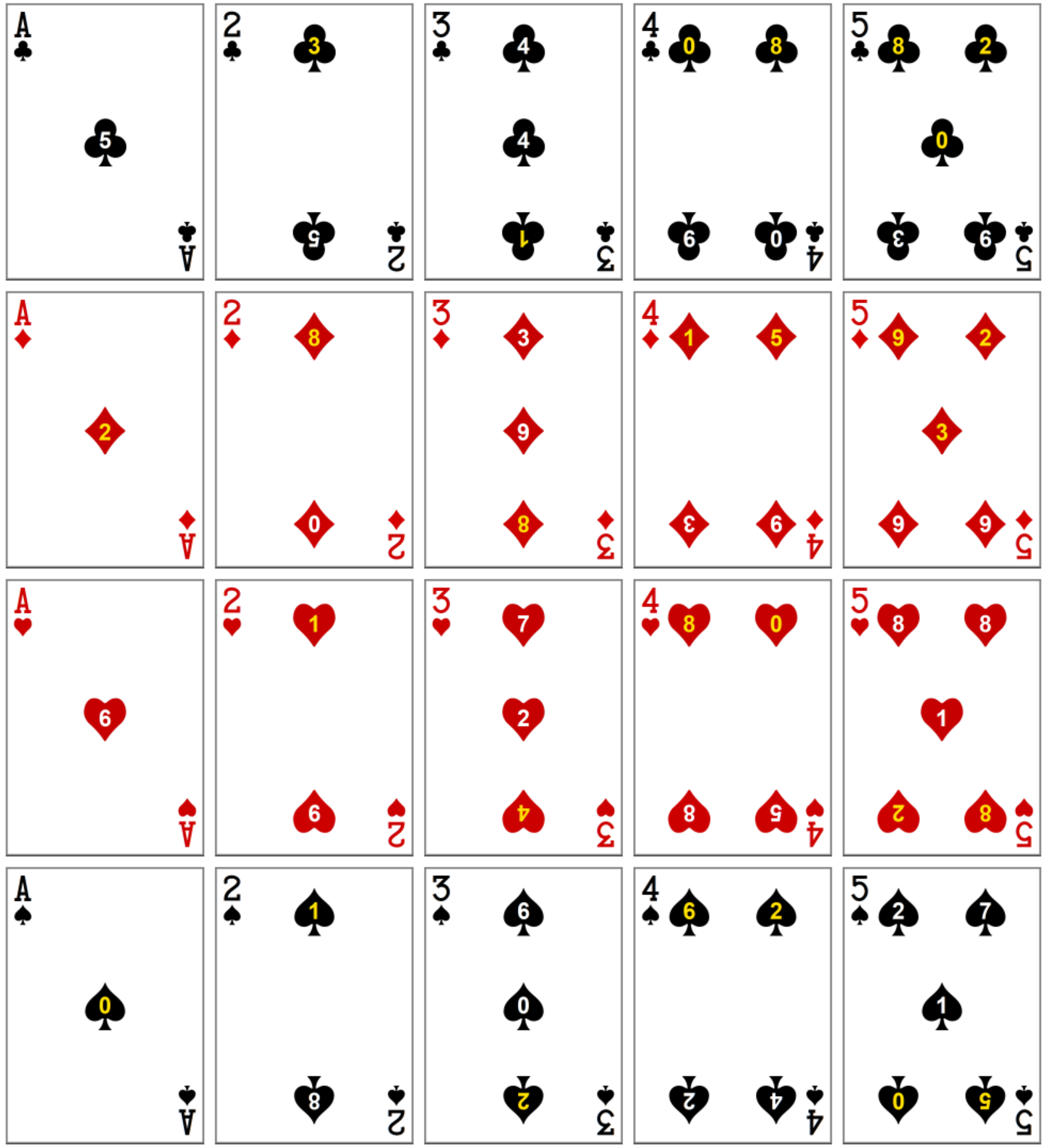
In this orientation the card is read as "2 8 4" (we read these digits in scan-line order, so it's "2" from the top line then "8 4" from the second line). Rotated, it reads as 9 2 5 9. Again there is the concept of a first and second deal for the face cards, with yellow digits facing up in the first deal and white in the second, and we denote by f_k the number of digits contributed by all the face cards in the *k*th deal. Since we allow 0 to 4 digits in each large pip, each f_k is in the range 0 to 48, with the 48 achieved when a deal has 4 digits in the upper left corner of all 12 face cards.

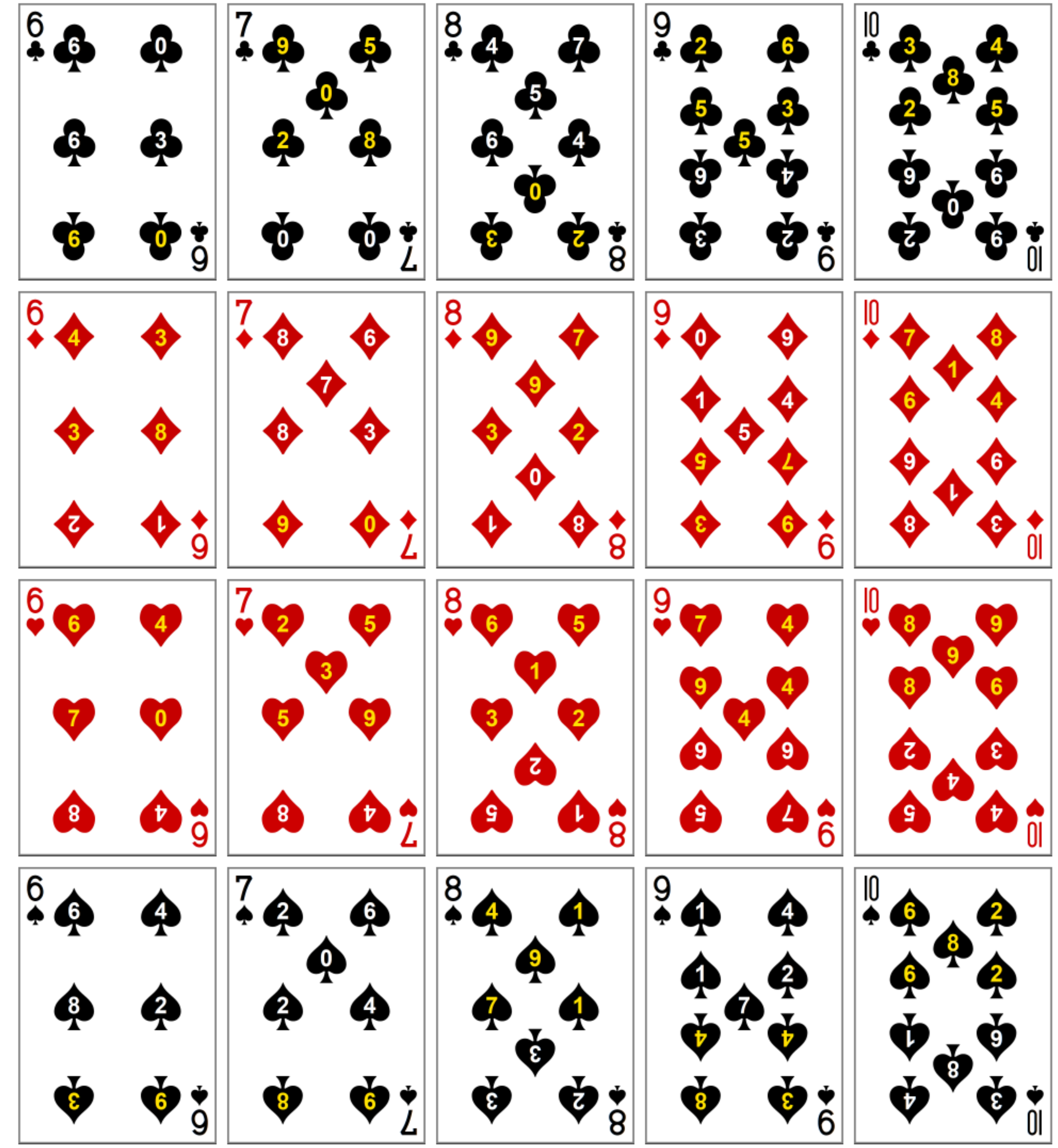
A *full deal* consists of laying out all 40 number cards followed by the 12 face cards, with all digits of one color facing up. The *k*th full deal ($k = 1$ or 2) produces a total of $n_k + f_k$ digits, and both deals together generate a total of $T = n_1 + f_1 + n_2 + f_2 = 300 + f_1 + f_2$ digits. Because $0 \leq f_k \leq 48$, T ranges between 300 and 396, depending on how many digits are inscribed on the face cards in each of the two deals.

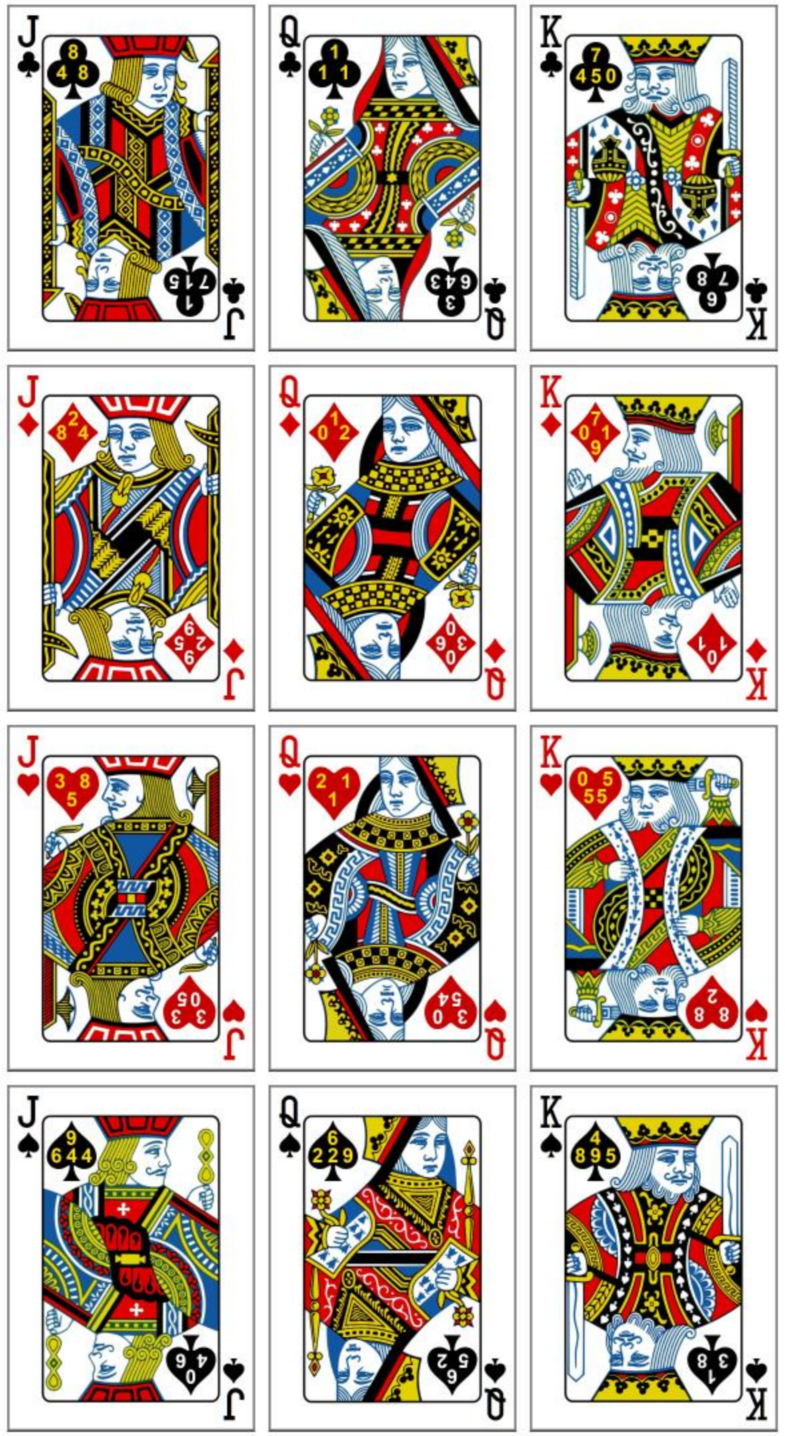
Puzzle statement. Take a deck of cards and label each of the 220 pips of the 40 number cards with a single digit of your choice, with the orientation of the digits matching the orientation of the pips, and with the two digit orientations on each card colored yellow and white as described above. On the asymmetric cards you can choose which "half" (*r* side or *u* side) gets the yellow numbers, then use white for the other half. Also inscribe 0 to 4 digits in each of the two large pips of each of the 12 face cards.

Make the first full deal of the 40 number cards (in any order) followed by the 12 face cards (also in any order), with the yellow numbers facing up on every card. Read off the index number and the yellow digits of each number card in order, and the yellow digits on the face cards, and write them all in sequence. Gather up the cards and rotate the whole deck by 180 degrees so that the white numbers are on top, and again order the number and face cards any way you wish. Deal the second full deal of number cards followed by face cards. Read off all the digits again and concatenate them to the first long digit sequence. The result is a sequence of 300 to 396 digits generated by two deals from the same deck of cards. The puzzle is:

> **Can we find a two-color labeling of a 52-card deck as described above, and an ordering for the first and second deal, so that the two deals generate a pre-specified sequence of digits, such as, say, <u>the first 300+ digits of the number π</u>?**

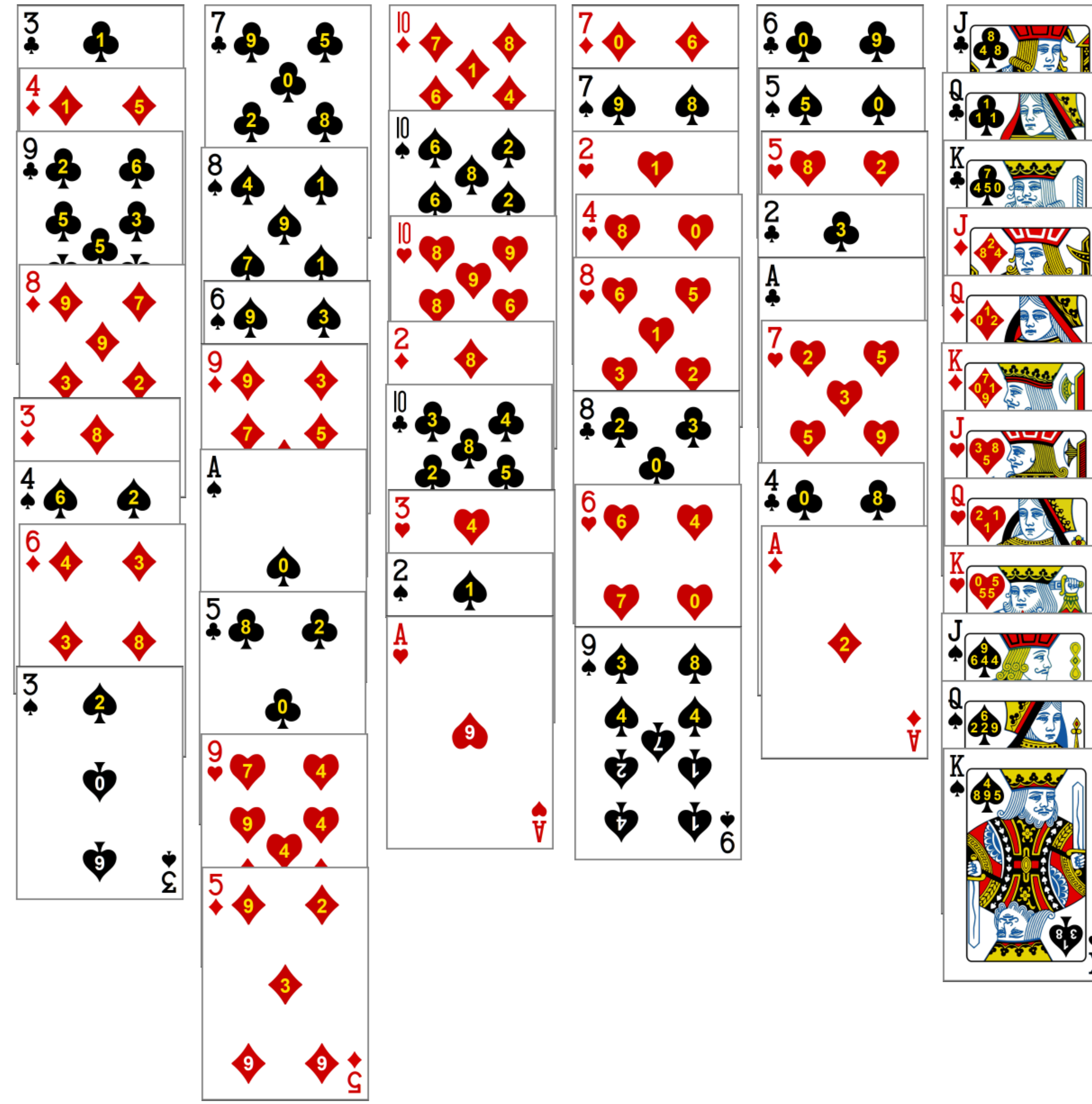






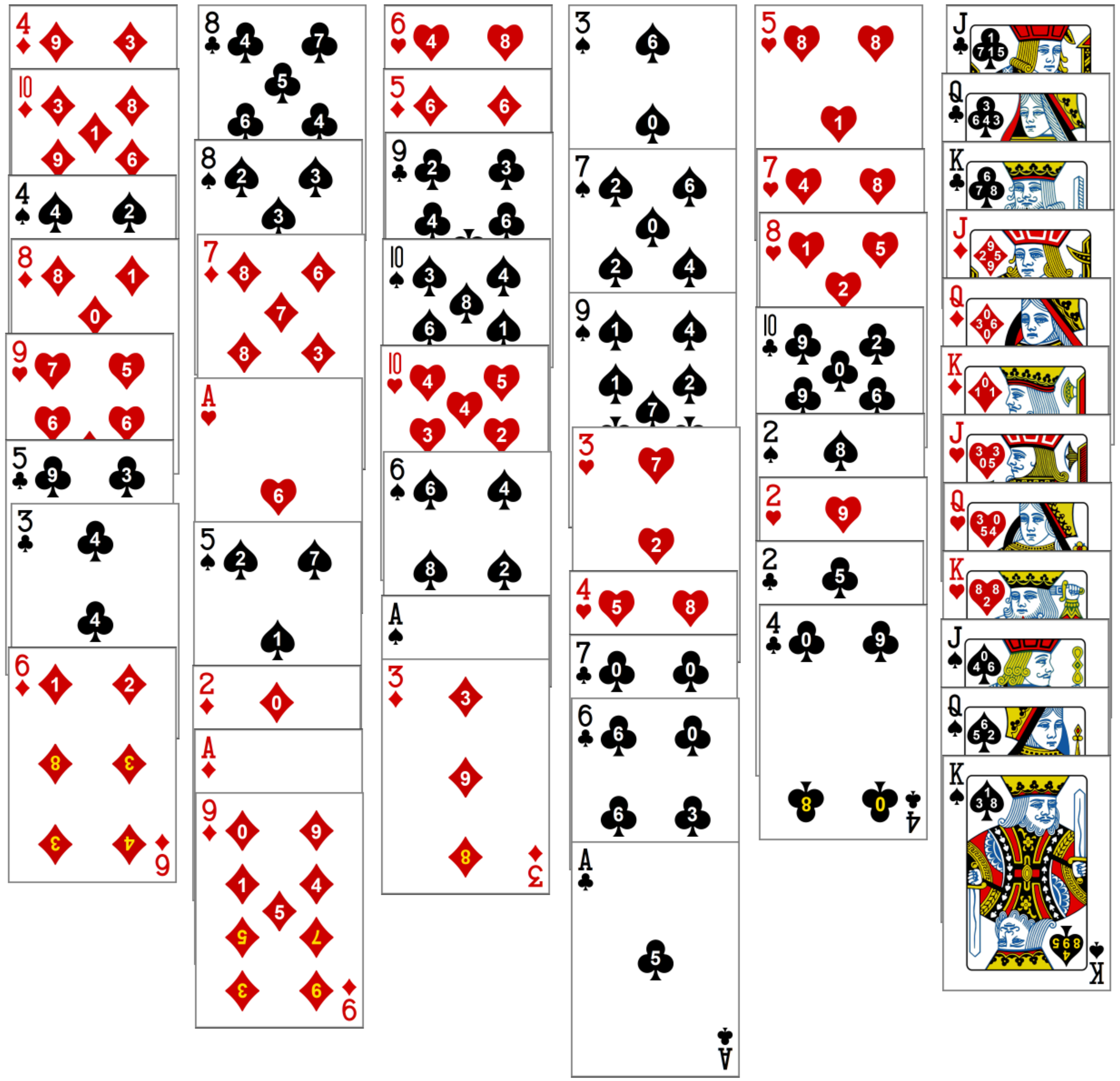
The next two pages show both deals: the first deal with all yellow numbers facing up, the second with white numbers. By dealing the cards in columns we can hide all the upside-down pips (except on the card at the bottom of each column) and directly read off the digits of π by just scanning down each column starting with the left one. The actual digits of π are displayed at the bottom of each page. Amazingly, this solution is perfectly balanced, with 150 number-card digits and 42 face-card digits in both deals, and it the *unique solution* with this property.

First Deal



Number cards 150 digits	31415926535897932384626433832795028841971693993751 05820974944592307816406286208998628034825342117067 98214808651328230664709384460955058223172535940812
Face cards 42 digits	848111745028410270193852110555964462294895

Second Deal



Number cards 150 digits	49303819644288109756659334461284756482337867831652 71201909145648566923460348610454326648213393607260 24914127372458700660631558817488152092096282925409
Face cards 42 digits	171536436789259036001133053054882046652138

Finding solutions. Suppose we try to find a solution to this puzzle by hand, and consider just the number card deals. The first card of the first deal has to be a "3", since its value must match the first digit of π. But now we have a choice of orienting this card with the two *r* pips facing up, which will get inscribed with the next two digits (1, 4), or with the one *u* pip facing up, which will get the single digit (1). In the first case the next card used must be an Ace, to capture the next "1", but in the second case the next card must be a 4, since the only digits captured so far are (3, 1). In general, both orientations have to be tried for every number card except for the rotationally symmetric 2's, 4's, and 10's, which is a total of 40 – 12 = 28 binary choices. During the second deal, the orientation of each card is further constrained by how the cards are oriented in the first deal, but quite a few still have to be tried in both orientations.

The number of branches in this search tree is too enormous for a search by hand, so we wrote a computer program that does an exhaustive, recursive, depth-first search for solutions for the first number-card deal, and then for each successful first deal does a similar search (whose starting point in the digits of π depends on the value of f_1) to see if the second number-card deal can also be constructed. The basic recursive task in this algorithm is to pick one more card from the deck, choose its orientation, add it to a tentative solution, then recurse. The value of this card must correspond to the next unused digit of π, but we usually have to try both orientations of the card, which determines whether it uses up the next r+1 or u+1 digits of π.

Recall that a 384-card deck has $36 \leq f_k \leq 48$, so there are 13 different choices available for f_1. We try all of these values in the order 42, 41, 43, 40, 44, etc., and stop at the first solution (if any) found by the algorithm described above. This finds a solution that's as close to balanced ($f_1 = 42$) as possible.

After success with the first 384 digits of π we ran the same search using the next chunk of 384 digits in π (i.e., digits 385 to 768). We found solutions for f_1 = 37, 41, 45, and 47 combined with n_1 = 147 or 149. While this is interesting, neither the number cards nor face cards are balanced in these solutions. We're greedy, and wanted a solution for the second deck that's perfectly balanced, so we wondered if there might be another degree of freedom we could use to help find a perfectly-balanced 384-digit deck for digits 385-768 of π.

Alternate Splits. There is, indeed, a subtle trick that can be employed to significantly enlarge the search space for finding solutions. Recall that the split into rightside-up and upside-down pips on each number card is determined by the traditional orientation of the pips in a standard deck, as shown in the diagram on the first page of this paper. Since we insist that rightside-up digits always go on rightside-up pips, we must always follow the (r, u) splits as shown in the table on page 1.

Strictly speaking, however, this rule only needs to be followed for the club, heart, and spade cards, since their suit symbols have a concept of "rightside up". This is not the case for diamonds, whose symbol is invariant under a 180-degree rotation. So we could, in theory, split the rightside-up and upside-down pip numbers differently on the diamond cards, and this will not cause any unwanted appearances of an upside-down digit on a rightside-up pip. We call these alternate ways of dividing the pips on the diamond cards *alternate splits* (AS for short), as opposed to the *standard splits* defined by the pip orientations of a traditional deck of cards.

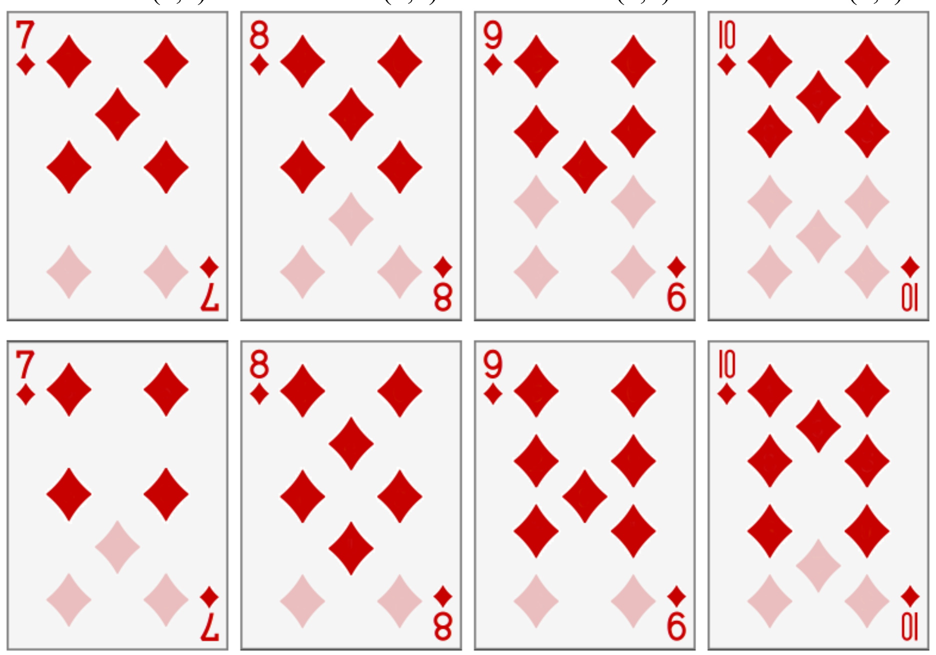
What alternate splits are possible? For aesthetic reasons, we insist on these two rules:

(1) The pips must be split by a horizontal line that runs the full width of a card. This means, for example, that 4 cannot be split as (3, 1).

(2) Both numbers in the split must be nonzero. So, for instance, 4 cannot be split as (4, 0).

These two rules mean that there aren't any alternate splits available for the A, 2, 3, 4, 5, and 6 cards, but the 7, 8, 9, and 10 cards do have alternate splits, as shown below. The cards in the top row are shaded to show the standard split, with the alternate versions depicted on the second row.

Standard: (5,2) Standard: (5,3) Standard: (5,4) Standard: (5,5)

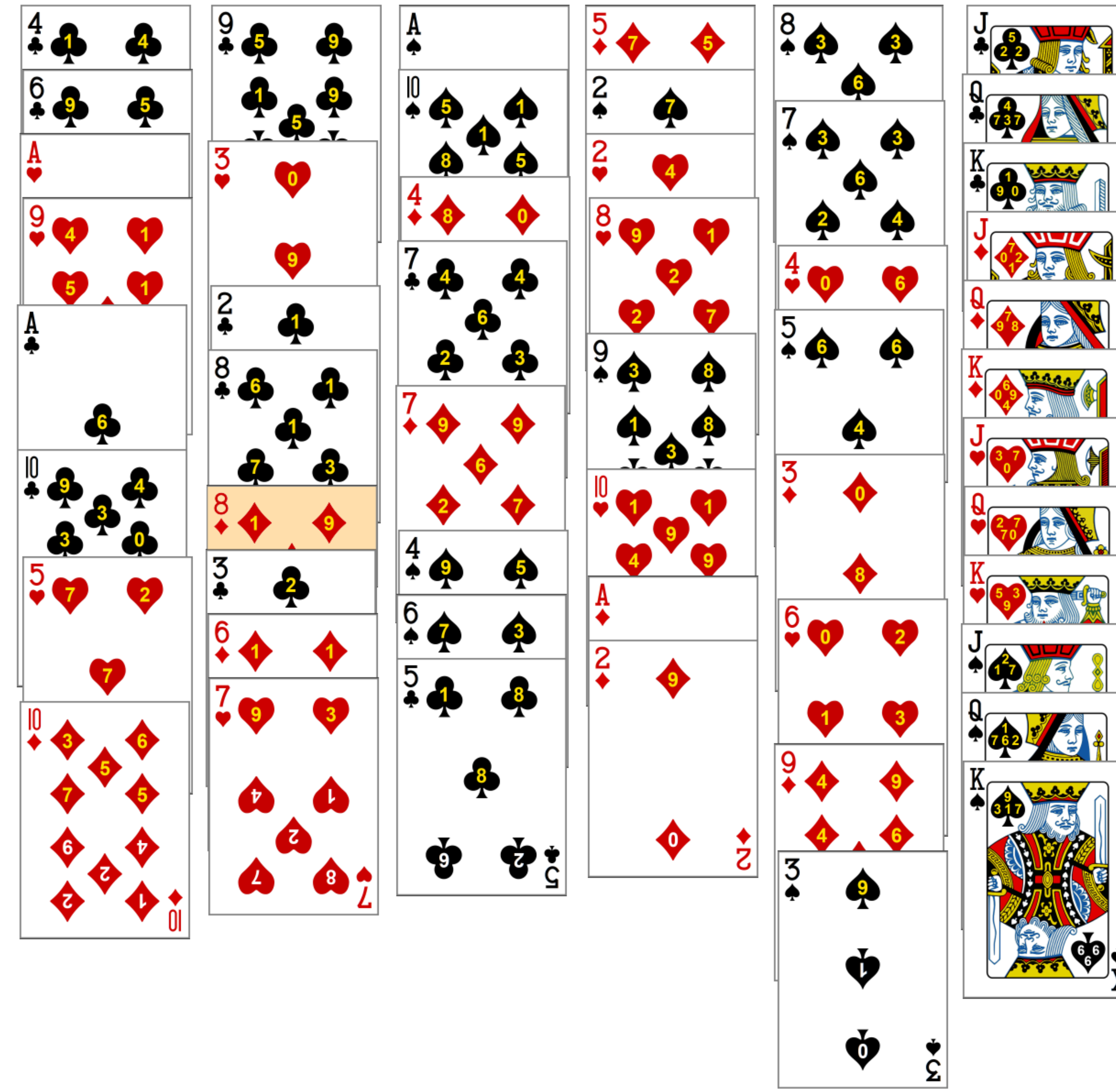


Alternate: (4,3) Alternate: (6,2) Alternate: (7,2) Alternate: (7,3)

There's actually a second split available for 10, which is (8, 2). We decided to not allow this one, as it is simpler and cleaner to have a single AS choice for each value (7, 8, 9, and 10). This means that there are exactly four cards in the whole 40-card deck (the 7, 8, 9, and 10 of diamonds) on which a unique alternate split can be used, if desired, to help achieve the successful construction of a solution. Since we can either use or not use the AS version of each of these four cards, there are 16 different AS configurations. So the full process of finding a solution now is to run the exhaustive search described above for each of these 16 AS choices.

We say that a solution using zero AS cards is *pure*. As already mentioned, there is no pure, perfectly-balanced solution for digits 385-768 of π, but there *are* some perfectly balanced solutions using AS cards. The two solutions with the fewest AS cards use just one: either the 8 or 10 of diamonds. The 8-of-diamonds solution is shown on the next two pages. (To save space only the two deals are shown, since the full numbering of all 52 cards can be inferred from the two deals.) The alternate 8 of diamonds, with its (6,2) split, is colored beige to make it easy to spot in both deals.

Second deck, First Deal:



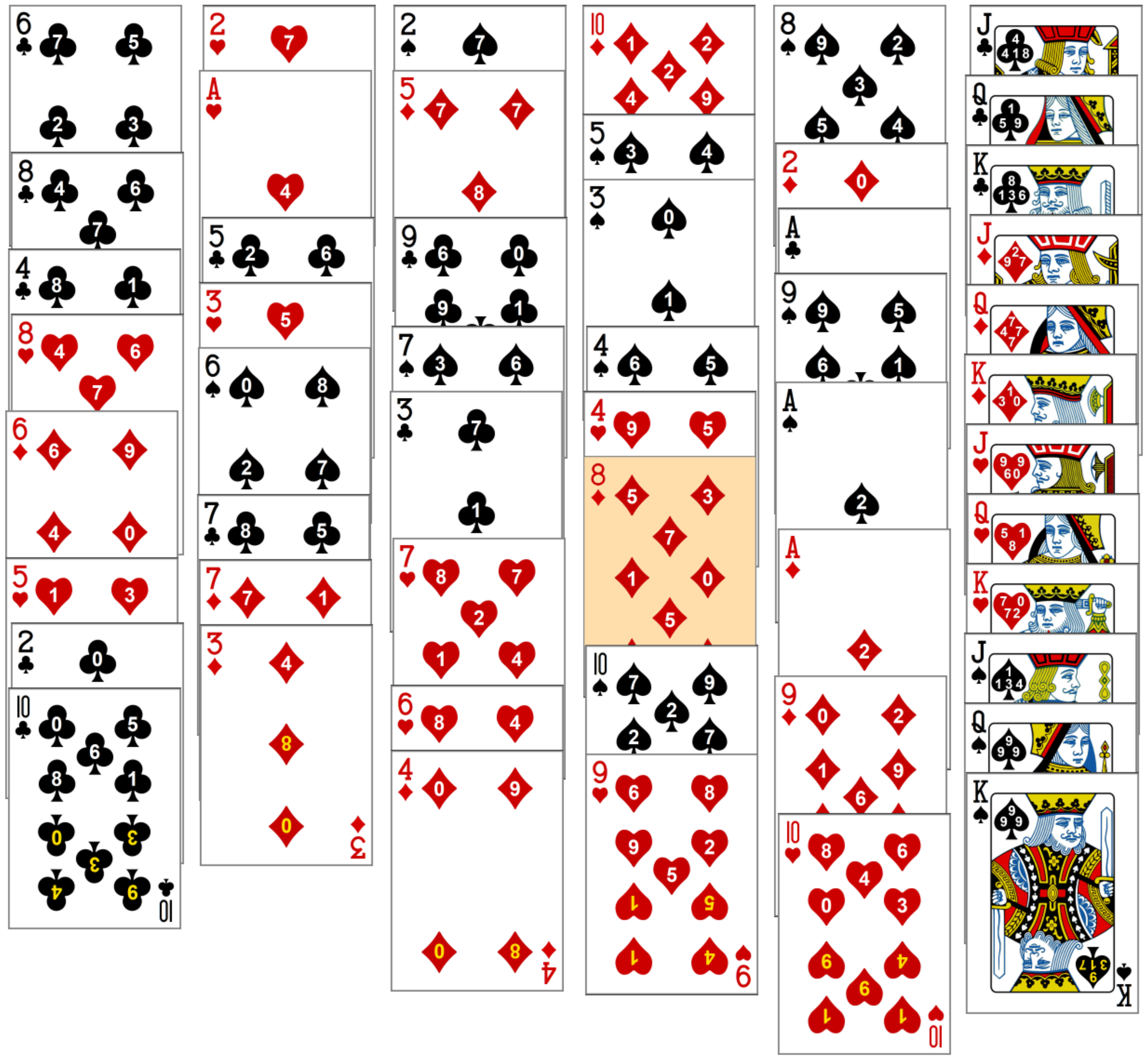
Number cards
150 digits

414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912983367336244065664308602139494639

Face cards
42 digits

522473719070217986094370277053921717629317

Second deck, Second Deal:



Number cards 150 digits	675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227968925892354201995611212902196086403
Face cards 42 digits	441815981362977477130996051870721134999999

Note the 999 and 999 on the final two face cards, encapsulating the famous 999999.

Continuing through the digits. What happens if we keep marching through the digits of π in 384-digit chunks? Can we always find a solution, or do some 384-digit groups occur for which no solution exists for any value of f_1 (in range 36 to 48) and any set of AS cards? To get some idea of what happens we looked for solutions for the first 261 384-digit chunks of π spanning 261 x 384 = 100,224 digits, which took roughly 200 minutes of runtime on a single core of a 10-core 2022-era PC (or 20 minutes using all 10 cores in parallel). Within this range, there are four places (starting at digits 14208, 38400, 57216, and 88704) where no solution exists. Is there some way to handle these troublesome spots?

Recall that these 384-digit decks have these restrictions:

[a] $n_1 + n_2 = 300$
[b] $f_1 + f_2 = 84$
[c] $36 \le f_k \le 48$,

We do not want to give up condition [a], but conditions [b] and [c] can be relaxed to:

[b’] $0 \le f_1 + f_2 \le 96$.
[c’] $0 \le f_k \le 48$

which was our original formulation, with a deck spanning from 300 to 396 digits, prior to fixing the deck size at 384 digits. So let’s distinguish between a *384 deck*, satisfying [a], [b], [c], and a *general deck* satisfying [a], [b’], [c’].

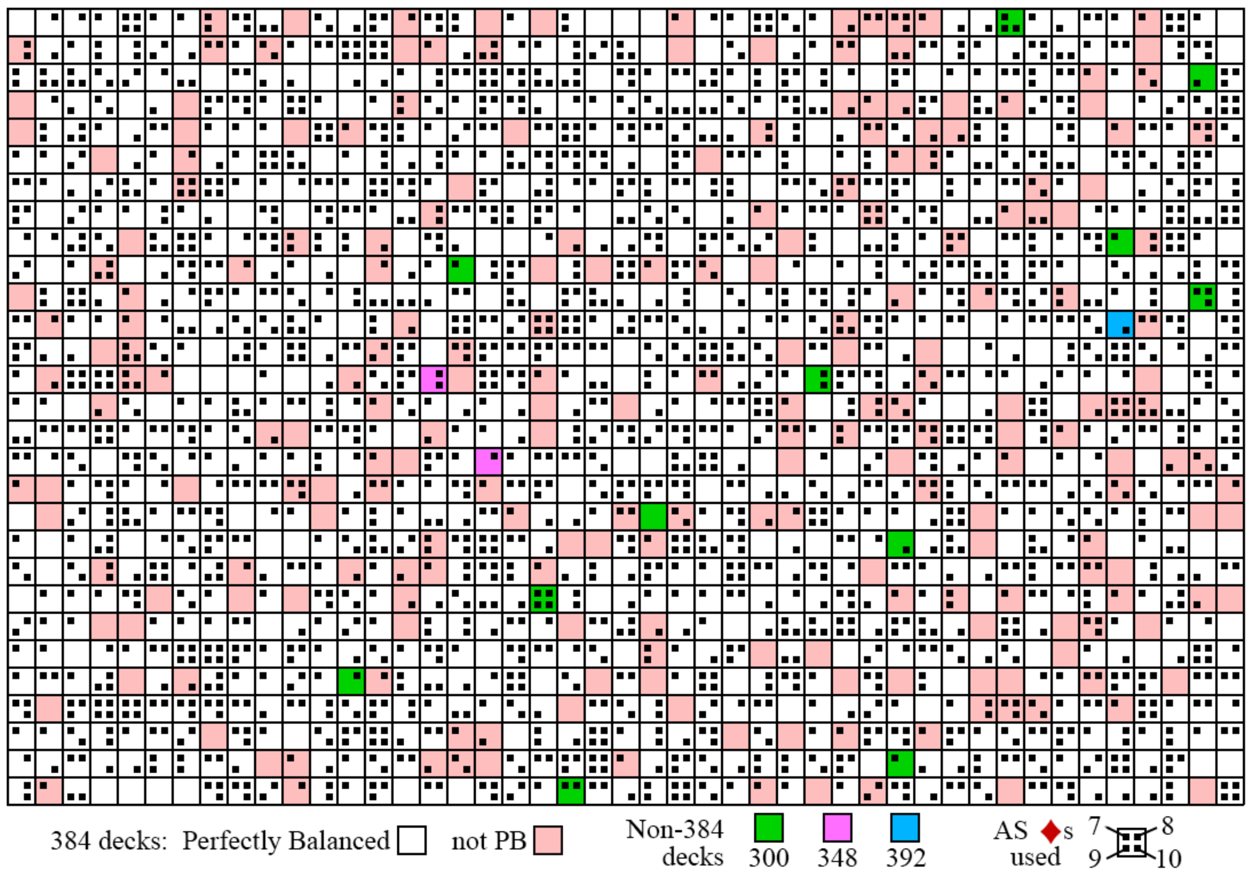
A possible strategy for getting past impossible positions in the digits of π is:

(1) Use 384 decks by default, marching through the digits 384 at a time.
(2) When a position p is reached where no 384 deck works,
- (2a) Move back to position p – 384.
- (2b) Try general decks with various values of f_1 and f_2 to (hopefully) find a solution. Let d (*not* equal to 384) be the number of digits in a deck that works here.
- (2c) Continue from position p – 384 + d using 384 decks.

Since $d \ne 384$, $p - 384 + d \ne p$, so in step (2c) we’re trying to find a 384-deck solution in a position different from the position, p, where the 384 deck failed in step (2). By trying various values of d in step (2b) we can hopefully find one that makes step (2c) work. There are 96 different values of d that can be tried: every integer from 300 to 396 except 384.

Using this strategy we were able to find a series of decks that encode the first half million digits of π (500,048, to be exact). In step (2b) we first tried a 300-digit deck (which basically dispenses with the face cards), since this provides the largest shift in position (-84) within the digit sequence, by which we’re hoping to overcome the “bad” position p. If a 300-digit deck didn’t work we next tried 348, then 392. Within this 500,048-digit range we never needed to try other values, meaning that we only used 3 of the 96 d values available.

The figure below shows the result of our 500,048-digit search, where each block represents one deck in a series of 45 x 29 = 1305 decks. For each deck we first attempted to find a perfectly-balanced (PB) solution, while also minimizing the number of AS cards. If no such solution exists then we looked for a non-PB solution. To distinguish them, perfectly-balanced 384 decks are colored white, while non-PB 384 decks are pale red.



Only 15 non-384 decks (of just three varieties) were required. These are colored green, magenta, or blue, as shown in the legend, according to their digit count. Although not indicated in the figure, 14 of these 15 solutions are perfectly balanced, the only exception being the single 392-digit deck. Overall, 1078 (82.6%) decks are perfectly balanced, and 125 (9.6%) of them are perfectly balanced *and* pure – including, as we have already mentioned, the very first one.

The AS cards used in a solution, if any, are indicated by one to four dots inside the square. The figure at the right in the legend shows which dot positions represent the 7, 8, 9, and 10 cards.

All of π? Does an infinite sequence of contiguous labeled decks exist with which to spell out *all* the digits of π? If the π-is-normal conjecture is true, the answer is no:

> **Theorem:** *If π is normal, any contiguous sequence of general decks (each spanning at least 300 and at most 392 digits) will eventually fail – i.e., will encounter a section of π's digits where no general deck can be constructed.*

Proof: Define a *bad window* of digits in π as a contiguous block of *n* digits of π which has the following property: The 40 number cards in a deck cannot be labeled in a way that allows either deal from this deck to capture the digits contained in the given window. One example of a bad window is *a block of digits in which any one specific decimal digit appears fewer than four times*. This works because there are four cards with each index number in the deck, and for each of these cards there must be digit in the window to assign it to. So if any digit occurs fewer than four times, a deal cannot be constructed.

Now recall that each deck has the structure $N_1 – F_1 – N_2 – F_2$, where N and F represent a block of number cards and face cards, respectively. The number of digits that can be captured by each N and F is bounded in size, so if the length, *n*, of the bad window is large enough then either N_1 or N_2 of *some* deck (in the sequence of decks) must lie entirely within the bad window, and cannot be constructed. The theorem follows by noting that if π is normal, arbitrarily large bad windows of the type described above are guaranteed to exist. ■

Bad window size. Exactly how large does the bad window need to be? The most digits that a number-card deal can capture occurs when the *r* pips of *every* number card are used in the deal. The sum of the ten *r* numbers (for A to 10) is 1 + 1 + 2 + 2 + 3 + 4 + 5 + 5 + 5 + 5 = 33, which multiplied by 4 for the four suits gives 132, plus 40 for the index numbers = 172. But we can increase this a little more by using the alternate split on the 8, 9, and 10 of diamonds (but not the 7, since the alternate split actually reduces the value of *r*); this changes the final 5 + 5 + 5 in the sum to 6 + 7 + 7, for a total of 177.

Now consider two number card deals with a face card deal between them (N – F – N). The most digits this can represent is when both N's are 177 and F = 48, so 177 + 48 + 177 = 402. If the bad window is one smaller than this (401), then no matter how the 402 digits of N – F – N line up with it, *at least one N will lie totally within it*, and therefore be impossible to construct.

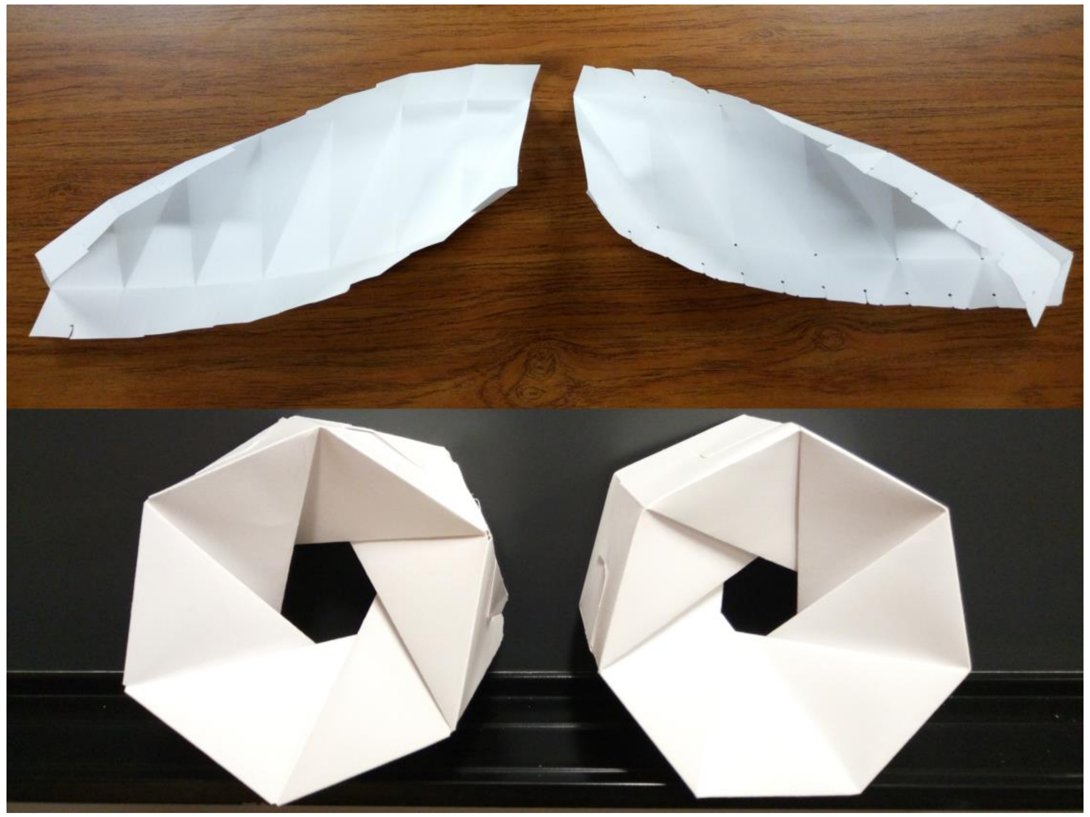
The location of a specific 401-digit bad window in π is not yet known. We searched the first 100,000,000 digits and found that the longest one is this 264-digit specimen at digit 6,562,558:

55219456178142178562058161430560084829194894522917
65224987912952876682978117724669017646018271765886
51349759408824181279876983955661018207966027682609
69925986952754875228992744105286487475109745400419
66491666472167120896527642127106288745970106469107
72458186210661

Note the three 3's shown in red, the only 3's in this whole group of digits. Also note that 264 is still a long way from 401! The question of how far we can continue this deck-building game in the digits of π before provably getting stuck remains an open problem.

Paper Tori

Alba Málaga Sabogal, Samuel Lelièvre & Pierre Arnoux



A flat torus is classically described as the geometric space obtained by gluing the parallel sides of a square, with the same orientation. One can start instead from any parallelogram, so that there is a whole family of flat tori. In mathematical terms, one can think of modding out the plane by the group generated by two translations.

The family of flat tori obtained this way has itself a rich geometric structure. It is an orbifold and it is called the modular curve (of tori). It is smooth almost everywhere except for two cone points corresponding to the square torus and the torus glued out from the 60-degree-angled parallelogram.

The precise mathematical construction of the gluing of the flat tori poses no particular issue. However, it is trickier to realize an isometric embedding of such a flat torus in euclidean 3-space. Recent work of Borrelli et al., using Nash embedding theorem, show how to realise a C1 embedding, but this embedding has a smooth fractal structure which is difficult to realize in practice, especially in paper!

We are looking here for a different type of embedding, an origami embedding, i.e., a continuous, piecewise linear embedding of the flat torus into the euclidean 3-space. The very existence of a non-trivial embedding of this kind is not obvious at all (in fact, till 3 months ago, we thought that such a locally flat embedding did not exist).

In late 2019 at the "Illustrating Mathematics" semester in ICERM, we learned about two possible realizations of flat torus embeddings with paper folding.

One realization was explained to us by Henry Segerman who learnt about it from a dedicated page on the French website mathcurve.com, with origins that can be traced to a Russian paper by Burago and Zalgaller from 1996, translated into English in 1997. The method consists in starting with a regular polygonal prism, take only its external walls, and then connect top and bottom polygons with congruent triangles. We will call these embeddings, prism embeddings; and the embedded tori, prism tori.

Glen Whitney showed us that a similar construction is doable starting from an antiprism, giving rise to antiprism tori.

In the gift exchange, we propose to give away flat layouts for flat tori. Our layouts have been thoroughly tested. They contain some excess paper and flaps so that once folded, they stand stably on their own, even without glue.

We also want to contribute a short paper about flat tori to the book. This will include a short history of the paper flat tori concept, then explain the buildup of the layout and the set of points in the modular curve covered by our paper flat tori. The set of flat tori that can be realized as polyhedral flat tori is still not completely clear.

By reporting results from the Illustrating Mathematics program, this material is based upon work supported by the National Science Foundation under Grant No. DMS-1439786 and the Alfred P. Sloan Foundation award G-2019-11406 while the authors were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI.

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An Origami-inspired Adventure in Number Theory and Programming

Jeanine Meyer

Abstract: *This paper describes an origami-inspired adventure. It will be a personal story, with attention to my history with mathematics and games. This focus is on the Dollar Bill Rosette model, created by Paul Jackson and modified by Martin Kruskal. The folding procedure is significant mathematically in [at least] two ways. It starts off with an iterative procedure that improves an original estimate, that is, decreases the amount of error. The folding procedure works, that is, goes through all the intermediate values, for a known class of numbers: reptend primes base 2. I came upon this class using programming in Python and online research. My proof that the numbers that work with the folding procedure are indeed the reptend primes base 2 is included. I term this an instance of "number theory in the wild".*

1 Background

My father always liked mathematical games and puzzles and, as a consequence, I did, also, because it was what we did. The family subscribed to **Scientific American**. I learned about origami from an article in the *Mathematical Games* section by Martin Gardner that featured the flapping bird. Later I met Lillian Oppenheimer, who taught me the Business Card Frog; her daughter-in-law, Laura Kruskal, teacher and origami inventor; and, more or less accidently, several of Lillian's grandchildren. I studied mathematics and computer science and worked at IBM Research in robotics and manufacturing research. When I came to academia, I used games and origami in my teaching examples. The model featured here inspired a book, **Origami with Explanations**, scheduled for publication Summer, 2020.

Several years ago, Mark Kennedy, master folder and teacher, organized informal folding events to take place while in line for New York City's Shakespeare in the Park. I have forgotten what play I saw, but one year I learned the dollar bill rosette. The model taught was the 22-panel rosette by Martin Kruskal.

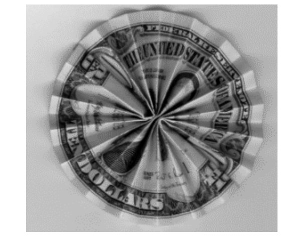


Figure 1: Dollar Bill Rosette.

At some point after learning the model, I showed it to a mathematician colleague at IBM Research and he said that he guessed that the numbers for which the procedure worked were a certain known class of primes. I now explain the folding of the model, which starts with making an estimate; show how the procedure improves the estimate; and then describe how I identified the class of primes using a program written in Python. I provide a proof connecting the class definition with the folding procedure. Lastly, I describe how this work inspired a new course and then a book project.

Note: I did not know Martin Kruskal (son of Lillian, husband of Laura, and mother of Clyde, one of my office mates when we were graduate students at NYU and highly regarded mathematician), but I would guess that he knew the mathematics, which I figured out on my own and will explain here.

2 Folding the Model

The first and main task in folding the rosette is to produce a fan consisting of 10 valley folds and 11 mountain folds. (If you want to do the Paul Jackson model, make 8 valley folds and then make a fan by putting mountain folds in-between the valley folds.) The valley folds divide the bill into 11 parts.

How do we make these 10 folds? First, estimate where the 1/11 position is on the dollar bill. The estimate is marked by putting a pinch on the side. Thinking of the pinch or mark as dividing the bill into 1-part and 10-part regions, we then divide the 10-part region in half by folding the end to the first mark, the one where we estimated one eleventh, and making a second pinch. The new pinch divides the bill into 6-part and 5-part regions. The pattern to note here is that there always will be two numbers, adding up to 11, with one even and one odd.

The next step is to divide (halve) the even portion of the two parts, setting a number N and set the other part to 11-N. This is repeated until you get back to 1 and 10. The sequence is

1 and 10
6 and 5
3 and 8
7 and 4

9 and 2
10 and 1
5 and 6
8 and 3
4 and 7
2 and 9
1 and 10

The last step produce a mark at the 1/11 area of the bill. Presumably it is close to the original mark. In fact, it will be an improved estimate! Yes, this is counter-intuitive. In the next section, I will explain how this process has improved the estimate.

Not counting the last row of the list, you see that 10 marks have been made on the dollar bill, all the intermediate positions. The dollar bill is divided into 11 equal size parts. Now, I will [quickly] describe the rest of the folding.

Going through the sequence again, we make full valley folds instead of pinches at each of the 10 positions. Next, make mountain folds in-between the valley folds to make a fan shape. These next steps are shown Figure 2: Completing the model. It works for the original 16 panel version as well as the 22-panel version. Divide the fan folds evenly into two parts and then unfold 3 segments on each side. Let's call these sections flaps. Fold the model over in the middle so that the two sets of flaps lie next to each other. Turn each combined flap into a tab shape and tuck each inside. Open up the fan to be a circle. The model is complete.

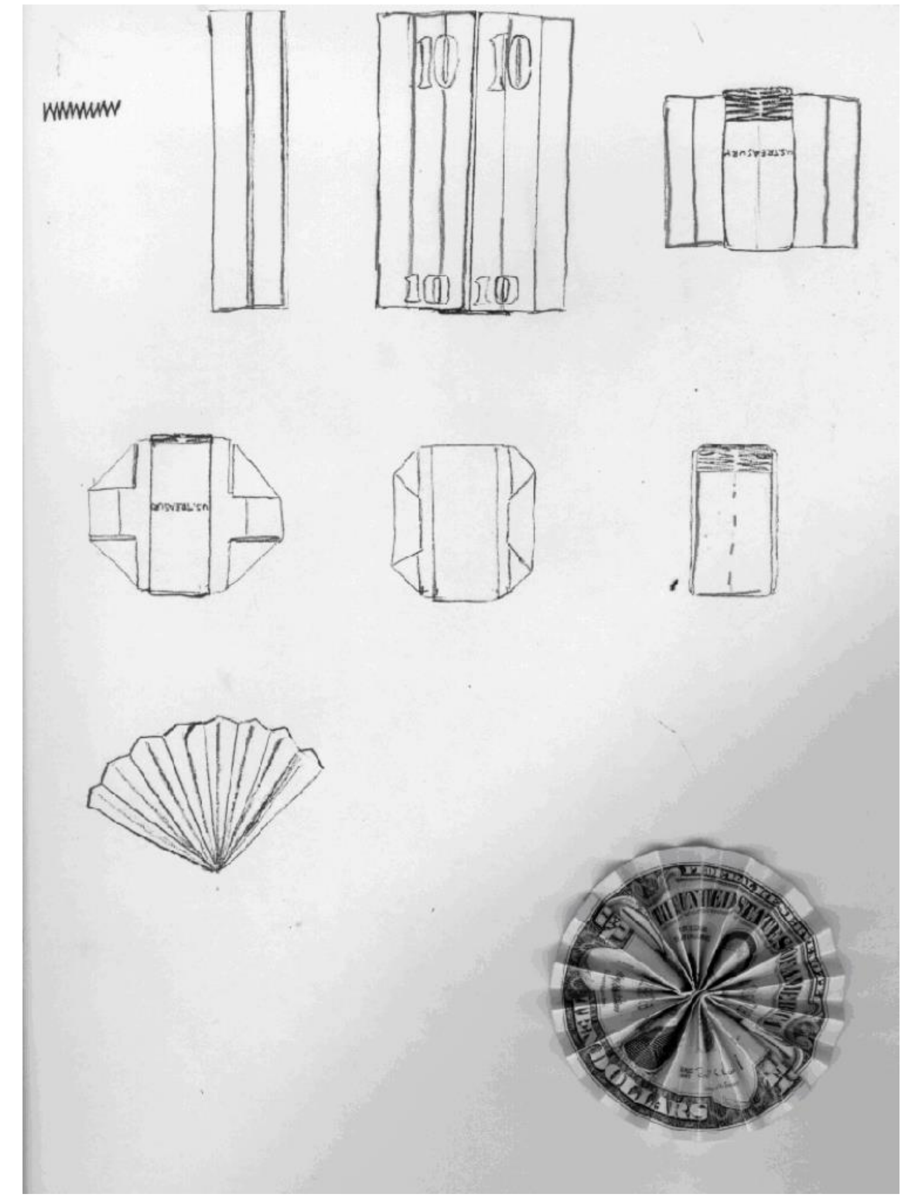


Figure 2: Completing the model.

3 Improving an Estimate

Most origami folders are familiar with what is termed the S method for dividing something into thirds. In the S method, you estimate what 1/3 would be and either make a mark or remember the position of your estimate. Orient the paper to look down on the edges and manipulate it into an S shape and then carefully turn the curves of the S into a Z and make the 3 parts the same size. A more systematic variation is to make a mark at what you think is one third on the edge of the paper. This divides the paper into one-part and two-part areas. Let's call the length of the edge L, the length of a true third t and where the mark was made t+e. The e is the amount of error. The length of the two-part area will be

L – (t+e).

(If you think of e as positive, this assumes that the original error was an over-estimate of a the third. The reasoning applies to an under-estimate.)

Fold the paper to the mark to divide the two-part area in two. We make the assumption that this fold is accurate. Make a mark. The distance from the edge to this second mark is half of L – (t+e).

(L-(t+e))/2

Making some re-arrangement of terms, this distance is

(L-t)/2 – e/2.

Since

t = L/3

Substituting for t we get

L/3 –e/2.

This shows that the error for the second mark is half of the original error. You can repeat the process is many times as you want to improve the estimate; that is, shrink the error.

The same phenomenon occurs when the rosette procedure is done. Assuming dividing a portion in half and the folding to a mark is accurate, the error amount is halved each time a section is divided into two parts. For the rosette model which involves 10 steps, the cumulative effect is to halve the error term 10 times! This means the original e is shrunk to $e/2^{10}$. The value of 2^{10} is 1024 so the final error is very small. To use mathematical language, the value of the error term e has limit zero. In practical terms, you can make it as small as you want.

3 The Number 11 and What Else

The next question relates to the number 11. The procedure of dividing the dollar bill edge into two parts and then dividing the even part in half goes on 10 times. Each of the intermediate points is hit. Does this work for all numbers? It does work for 3 but what other numbers?

The first observation is that the number must be odd so that any partition into two parts yields one odd and one even part.

NOTE: Following the practice in programming, I use the asterisk for multiplication.

A next observation is that the number must be prime. Consider the case of 15. The procedure would start with 1 and 14 and then continue as follows

1 and 14
8 and 7
4 and 11
2 and 13
1 and 14

We note that this sequence does not hit all the intermediate points. Here is an informal proof:

Suppose P is not prime, say it is equal to M * N, where M and N each >1. Note that neither M or N can be even.
In the rosette procedure, at some point, the sizes for the two portions must be M and (N-1)*M. What is the next step? M is not even, so the next step would be

M+((N-1)/2)*M and ((N-1)/2)*M

Continuing the folding process, each of the pair of numbers would have a factor of M. That is, it would not continue to a pair with one of the two equal to 1. One way to make this more concrete is to consider the number 9. In this case, M and N are each equal to 3. Assuming the process works, applying the folding procedure to 3 and 6 results in 6 and 3 and then 3 and 6. The procedure gets stuck and never reaches 1 and 8.

The procedure does not work for all primes. Consider the situation with 17. Here are the successive pairs produced when we start with 1 and 16. The procedure ends, that is, returns to 1 and 16, but does not hit all the intermediate points.

- 1 and 16
- 9 and 8
- 13 and 4
- 15 and 2
- 16 and 1
- 8 and 9
- 4 and 13
- 2 and 15
- 1 and 16

At this point, I recalled the conversation from many years ago that there is a certain class of primes that may correspond to those satisfying the origami procedure. I decided to investigate.

4 Write a program and Search the Web

Python is the language I used to check if numbers work using the folding procedure. I chose Python because it is the language we use in our Number Theory course, which makes use of a book, **Elementary Number Theory with Programming**. We made that decision because Python has arbitrary precision for integers.

[Aside: JavaScript is used in the book because the authors believed it to be an easier language for people less familiar with programming to follow. In contrast to most colleges, Purchase College/SUNY offers only a joint Mathematics/Computer Science major, so the students taking the Number Theory course have had at least one programming course. The Number Theory course provides us a way to introduce another programming language, Python. Our students can appreciate the advantage that the arbitrary precision provides for number theory and can appreciate when it is critical to avoid leaving the integer domain for floating point numbers.]

My Python program is shown below and is, hopefully, readable. It is my only Python program. Comments start with #. Indentation is required to indicate the content of functions and clauses. The function, `tryProcedure`, is invoked with a value `N` as argument. The variable `count` keeps track of the number of steps. The variables `currentpos` and `remainder` describe the pair of numbers (parts). The # symbol indicates a comment for the rest of the line.

```
def tryProcedure(N):
   count = 1          # start with 1 and N-1
    currentpos = 1
    remainder = N-currentpos
    while True:
      if (isEven(currentpos)): #  determine even side
           currentpos = currentpos//2
             # the // forces integer division
           remainder = N - currentpos
      else:
           currentpos = currentpos + remainder//2
           remainder = N-currentpos
     count = count + 1

     if (currentpos==1):   #at 1, leave loop
           break
# outside of the while loop
   if (count==N):
       print(" ",N,end="")   #This is a good value
   return
```

The operation of integer division, indicated by the //, is important for keeping everything integers.

The program prints out a good number, that is, the numbers that go through N steps before returning to the pair 1 and N-1. Invoking the `tryProcedure` function from 3 to 1000 produced the following list of numbers:

3 5 11 13 19 29 37 53 59 61 67 83 101 107 131 139 149 163 173 179 181 197 211 227 269 293 317 347 349 373 379 389 419 421 443 461 467 491 509 523 541 547 557 563 587 613 619 653 659 661 677 701 709 757 773 787 797 821 827 829 853 859 877 883 907 941 947

Since I did not remember what my colleague said some years ago, I attempted to consult the institutional memory of the web by putting this whole set of numbers into the Google search field. I was not that optimistic, but it was successful! I reached https://en.wikipedia.org/wiki/Full_reptend_prime

5 Proof the two Classes are the Same

[Note: Repeat: I do not claim to be the first person to prove that the folding procedure for the rosette and the reptend prime base 2 procedure are the same. I did not find a proof, but I did not look very hard because I liked thinking about it myself.]

The reptend prime base 2 class is defined as follows:

> A number P for which 2 raised to the power N, N going from 0 to P-2, produces the numbers 1 to P-1, modulo P, is a reptend prime base 2.

If the P-2 seems strange, do note that the process starts with 0, not 1.

Here is the reptend procedure for 11:

2^0 is 1 = 1 mod 11
2^1 is 2 = 2 mod 11
2^2 is 4 = 4 mod 11
2^3 is 8 = 8 mod 11
2^4 is 16 = 5 mod 11
2^5 is 32 = 10 mod 11
2^6 is 64 = 9 mod 11
2^7 is 128 = 7 mod 11

2^8 is 256 = 3 mod 11
2^9 is 512 = 6 mod 11
2^{10} is 1024 = 1 mod 11

This sequence, that is, the defining characteristic of reptend primes base 2, resembles the folding sequence for 11, but in reverse order. This certainly is not a proof since it is just the one number, 11, but it is encouraging.

To prove that the numbers that can work using the folding procedures are the reptend primes base 2, one needs to prove that the numbers for which the folding procedure hits all the intermediate numbers are the same as the numbers for which the reptend process, raising 2 to powers from 0 to the number -1, hits all the intermediate numbers. I decided to try for a stronger result: the two procedures are the same procedure, with the folding procedure done in reverse order. Proving the bigger thing seemed easier to me than proving the smaller thing. That is, if N and P-N are pairs in reverse folding, then I will show that

$N = 2^k \bmod P$

for all k steps starting from 0, for all primes P.

So how to define the reverse folding process? There are several ways to approach this challenge. If (F and P-F) goes to (G and P-G) in the normal folding procedure, I need to define F in terms of G. I can consider cases of if F was odd and if it were even. Instead, consider the following. Either F was halved or P-F was halved. So either F is equal to 2*G, or P-F is equal to 2* (P-G). Which one happened? The answer is to consider if 2*G is greater than P or not. Keep in mind that P is prime so 2*G cannot be equal to P. Also, since the pair of numbers, G and P-G add up to P, one is less than ½ of P and one is greater. So doubling one will be greater than P and doubling the other will be less.

Initial case: k=0

- Reptend and reverse folding start out with
  $2^0 = 1$, so $2^0 = 1 \bmod P$

Induction step

- Can assume $G = 2^k \bmod P$ meaning
  $G = 2^k + a*P$
- Two cases: 2*G< P and 2*G > P.
  Case 2*G< P.
  So F = 2*G. Substituting the expression for G
  $F=2*(2^k + a*P) = 2^{k+1} + 2*a*P$ so $F = 2^{k+1} \bmod P$

- Case 2*G>P
  F = P-2*(P-G)
  F = P – 2*P + 2*G Rearranging terms
  F = 2*G – P Substituting the expression for G
  $F = 2*(2^k + a*P) – P$
  $F = 2^{k+1} + 2*a*P – P$
  $F = 2^{k+1} + (2*a-1)*P$
  $F = 2^{k+1} \bmod P$

To recap: Because both processes yield the same results, both either satisfy both the reptend AND the folding criteria of hitting all the intermediate points between 1 and P-1 or neither do.

To put it another way, the sequences of numbers are the same starting at k = 0 and continuing for all integers! However, we only consider the values up to k = P – 2 for each P.

6 Reflection

This paper describes exploring an origami model, the dollar bill rosette. The model provided opportunities to touch on topics in basic algebra, limits, programming, and number theory. It also demonstrates what is a proof and the benefits and the limitations of web searches. A talk on this process, which we refer to as an *adventure in origami*, has been given several times to our Number Theory and Senior Seminar classes and the response from the students is strongly positive.

In fact, my chair, upon hearing about my adventure, suggested designing a general education course based on origami. As one of many colleges that require everyone to take a math class, we always are looking for new courses. My first reaction was that the mathematics associated with origami was too difficult for most students. However, late one night, I was inspired and came out with a plan, making use of origami models to inspire topics in basic algebra, geometry and trigonometry. For example, final dimensions of the model can be computed in terms of the size of the (flat) paper. Students can think about the change from 2D to 3D. We can compare crease patterns, folding sequences and final models.

The dollar bill rosette model is taken up after simpler dollar bill folds. I don't expect the students to understand every aspect. It does seem that most of the students in the two classes to date:

1) are initially surprised, but then understand how the procedure improves the initial estimate (the initial surprise is important)
2) accept that seeing that the numbers that work for the folding procedure match this specific class of primes up to 1000 does not prove that the two classes are the same; and
3) appreciate that my proof is stronger than just proving the definitions produce the same numbers…but sometimes stronger is easier.

In addition, I hope they observe for all the models my excitement and delight at the beauty, structure, and patterns of the origami and role of mathematics. The course also includes peeks at origami mathematics topics such as tessellations, flat-foldability and fold-and-cut. I now am working with a former student, now colleague and co-author, on a book project taking this approach. The models are traditional and modern, including action and modular models. Many models are made from squares, such as Japanese kami. However, inspired by Laura Kruskal, who favoured using so-called found paper, in addition to the dollar bill rosette, there are other dollar bill models, and also models from business cards and copier paper. See Figure 3 for some of the models. We appreciate the permissions granted by the designers and the general support. The adventure continues.

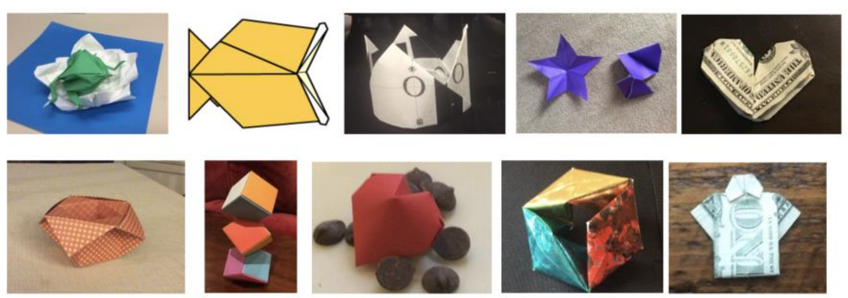


Figure 3: Selection from models used in course and book.

References

Full Reptend Prime, **Wikipedia**, https://en.wikipedia.org/wiki/Full_reptend_prime, last edited March 22, 2017.

Jeanine Meyer and Takashi Mukoda, **Origami with Explanations**, World Scientific Publishing, Summer/Fall 2020.

Lewinter, Marty and Jeanine Meyer, **Elementary Number Theory with Programming**, Wiley, June, 2015, http://www.wiley.com/WileyCDA/WileyTitle/productCd-1119062764,subjectCd-CS00.html

More Fun with Langford's Problem!

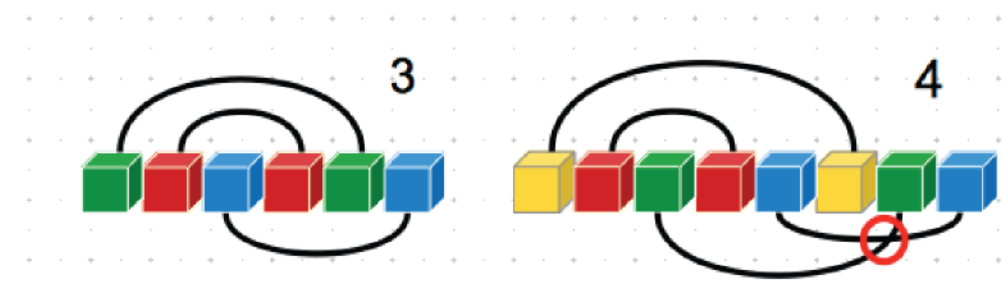
by John E Miller
Portland, Oregon, USA

Abstract

The known number of solutions to the classic problem has been extended through n=28;
Similar progress was made on Knuth-planar solutions;
An End Run variation of planar solutions has been defined and explored;
A subset of solutions of the classic problem conform to a Colombian Variant;
Tanton's Chairs - A puzzle based on a circle of chairs was explored as a variant of LP;
Langford Quilts. We provide exercises and a link to the Langford's Problem web page.

Introduction

Consider these pairs of colored blocks. There's one block between the red blocks, and so on.



If you try do this with five or six pairs, you won't succeed. It's not hard to prove that you can fit blocks together in this way only when the number of pairs is a multiple of four, or one less.

There are 26 unique ways 7 pairs of blocks can be arranged; 150 solutions with 8 pairs.

At G4G11 (2014), I reported that Langford's Problem had been enumerated up to 24 pairs of blocks.

In July 2015, Team Assarpour-Liu, extended the known number of solutions

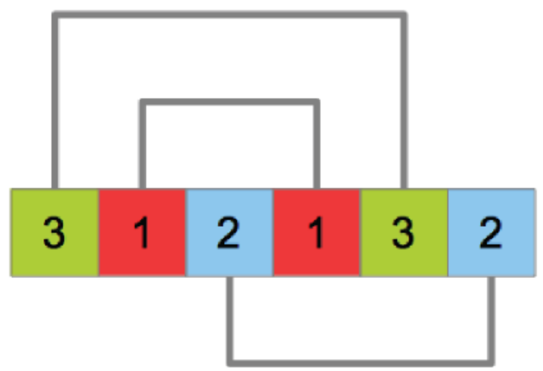
```
27 ==>   111,683,611,098,764,903,232    (111 Quintillion)
28 ==> 1,607,383,260,609,382,393,152    (1.6 Sextillion)
```

No solutions are possible for 29 & 30 pairs. Counts for 31 & 32 pairs are unknown.

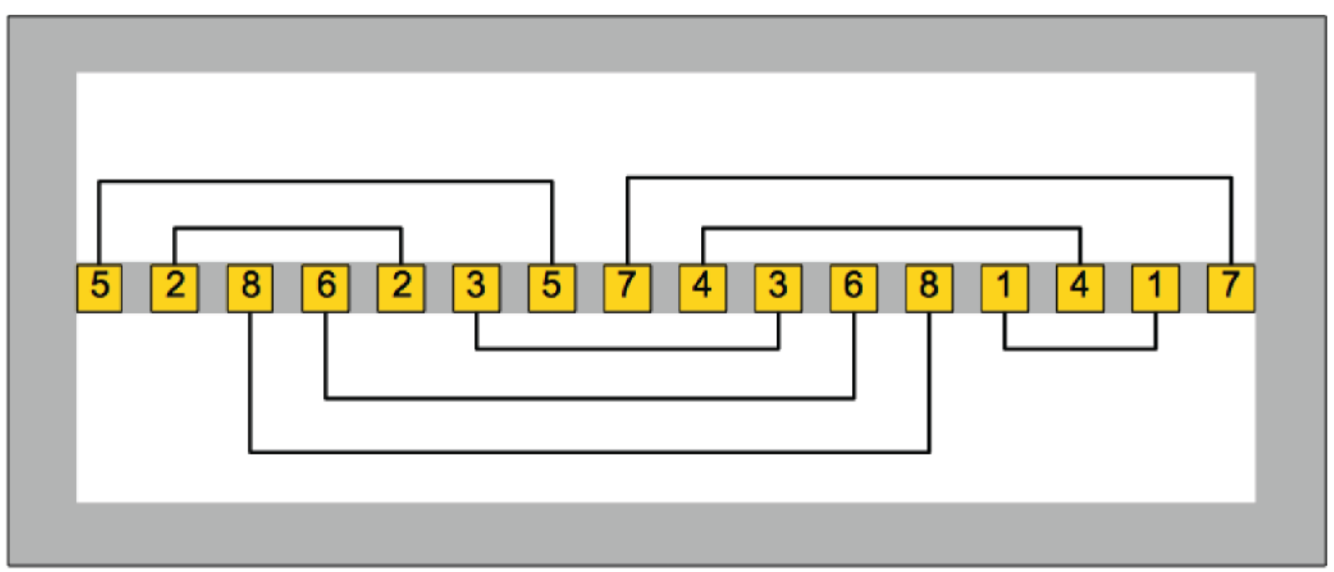
There is no combinatoric formula for the number of solutions. However, Dr Zan Pan recently developed an asymptotic formula estimating the number of Langford and the Skolem sequences! See the website for a link to Dr Pan's paper.

Planar Solutions

What is a Knuth-Planar solution? Pairs can be connected by lines in the plane, *without crossing*. By definition, it's 'no fair' going around the end of the arrangement to avoid crossing. Three pairs are naturally planar.



In the Introduction above, four pairs are not Knuth-planar. (See red circle around the crossing.) Think of connections only allowed in the white areas above and below the arrangement in this diagram of a planar solution for 8 pairs:



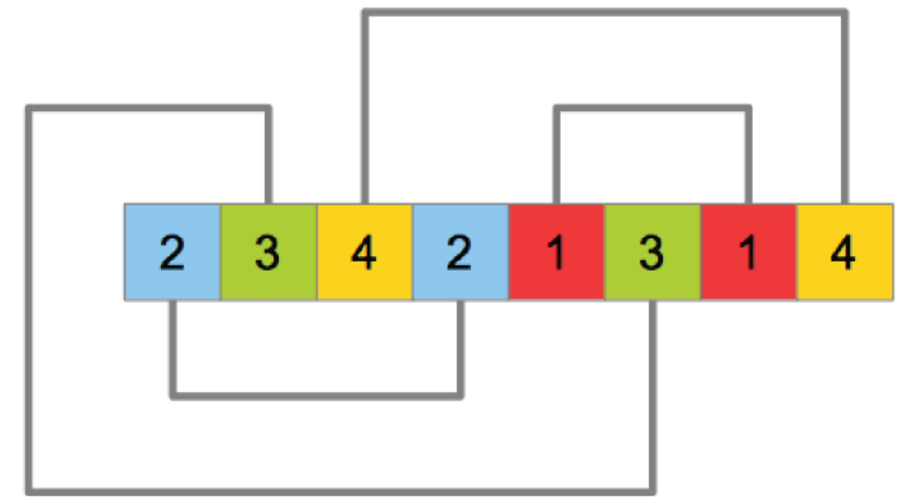
Knuth solved for Planar Solutions up to n=28 in 2007. In January, 2018, Rory Molinari confirmed Knuth's previous results, and obtained these results

P(2,31) = 5,724,640
P(2,32) = 10,838,471

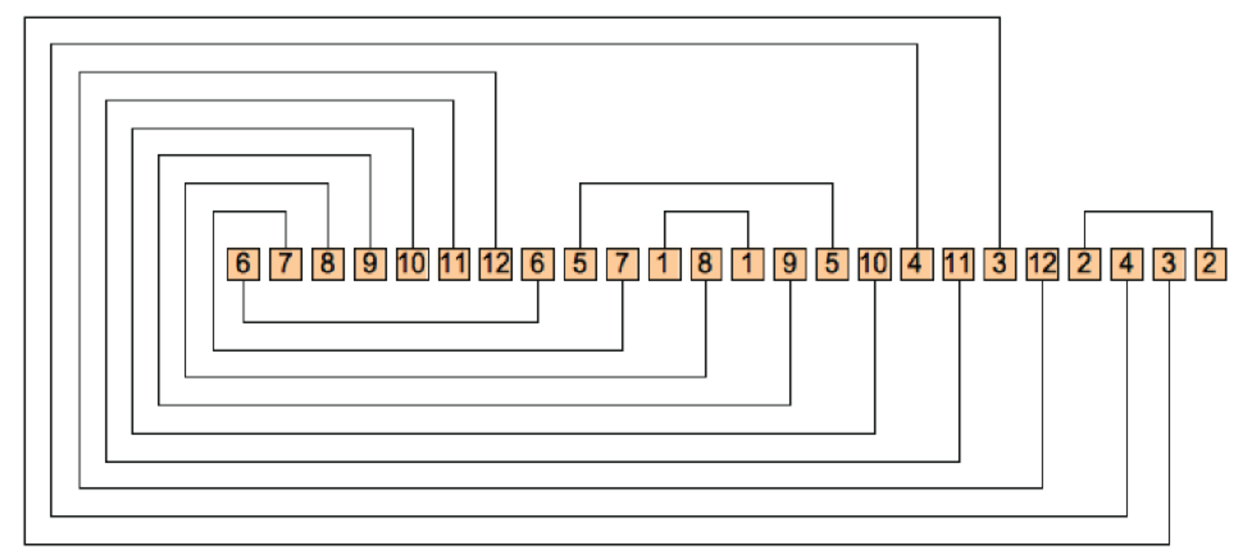
Computer programs were able to enumerate because the search space is radically reduced.

End Run Planar Solutions

Rory Molinari investigated the concept of End Run Planar Solutions. More arrangements can be classified as planar if allowed to do one or more end runs.



Look at all the Looping Joins! Molinari coind the term, and made assertions about them.

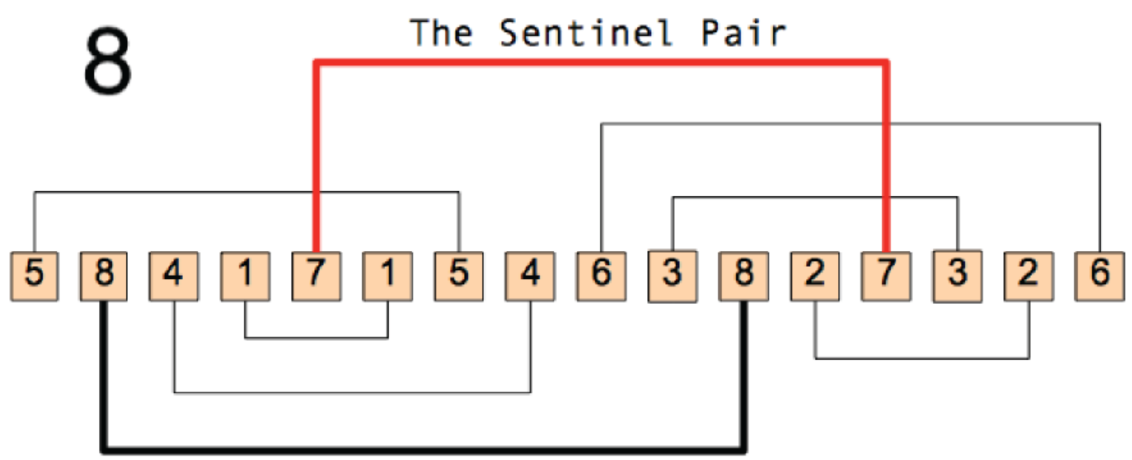


There is a page dedicated to Molinari's work on the Langford's Problem set of web pages.

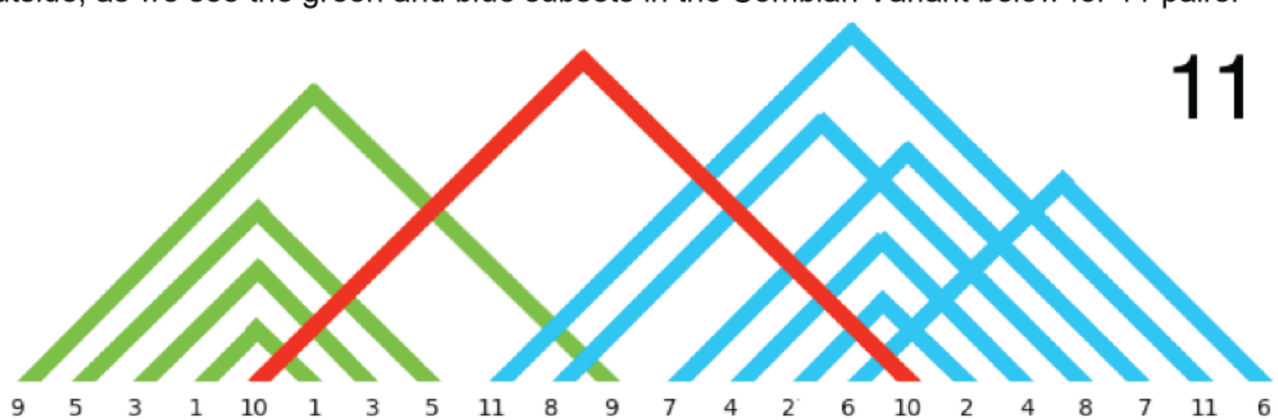
A Colombian Variant

This variant of Langford's Problem was cooked up in Colombia by Freddy and Bernardo in 2019.

A sentinel pair restricts all smaller pairs from being completely inside the sentinels.



Consider the above arrangement for 8. Put the 8's in an empty array, and then put the 7's in such that they partition the arrangement into left and right. Here we see 4 empty spots to the left and 3 on the right. The 8 takes one of the spaces on the left. The Colombian Variant constrains the remaining pairs to have one leg between the sentinel pair and its other leg outside, as we see the green and blue subsets in the Combian Variant below for 11 pairs.



Colombian Enumeration Very few solutions of LP conform to CV constraint.

Colombian Variant connects to the Davies construction method (1958), i.e., the singular Davies Construction results in a Colombian Variant solution.

The Colombian Variant has a dedicated page on the Langford's Problem set of web pages.

Summary of Numbers of various solutions

End Run Planar includes more solutions than Knuth-Planar, because you have the Knuth planar plus end run planar solutions.

	CLASSIC	PLANAR	END RUN	COLUMBIAN
3	1	1	1	1
4	1	0	1	1
7	26	0	6	3
8	150	4	24	10
11	17792	16	139	76
12	108144	40	289	140
15	39809640	194	2414	2478
16	326721800	274	4455	5454

There is a large Table of Solutions to befound on the Langford's Problem set of web pages.

Tanton's Chairs (A circular version of LP)

10 students sit in a circle. Is it possible for one student to move 1 place clockwise, one student 2 places clockwise, one student 3 places clockwise, and so on, all the way up to tenth student moving 10 places clockwise (back to his/her own seat) and ALSO end up one student per seat? 11 students?

To solve this, you must match up unique "future chairs" for the students with the constraint of also moving unique number of chair "units". Each student is destined to move a unique number of chairs clockwise, to (hopefully) claim an empty chair.

I recognized that Tanton's present-and-future chairs are like Langford's pairs of colored blocks. So I hacked my "standard algorithm" for Langford's Problem to use a wrap-around array and counted solutions for 1..15 students/chairs. ThesSolution sequence seems to be: 0, 0, 1, 0, 3, 0, 19, 0, 225, 0, 3441, 0, 79259, 0, 2424195, ... (Only odd numbers of seats work, so there are 0's in the sequence.) Tanton's question about 11 chairs has 3,441 solutions!

Due to the Circular nature, without loss of generality, we only need to examine solutions with the first student starting in a particular chair. (Otherwise, we'd duplicate solutions.)

Two sample solutions for n=11. Figures below show sample solutions for n=11, using straight arcs, rather than lines curving around the circle of chairs.. A short line that looks like it might be going counter-clockwise is just taking a direct path. Green denotes a Start chair; Reds are Stop chairs.

One figure shows two non-trivial cycles. The other figure shows a solution for 11 chairs, consisting of five pairs of students swapping chairs, and one staying put (the one looper).

There may be a better representation of Tanton's Chairs using circular arcs. Feel free to play!

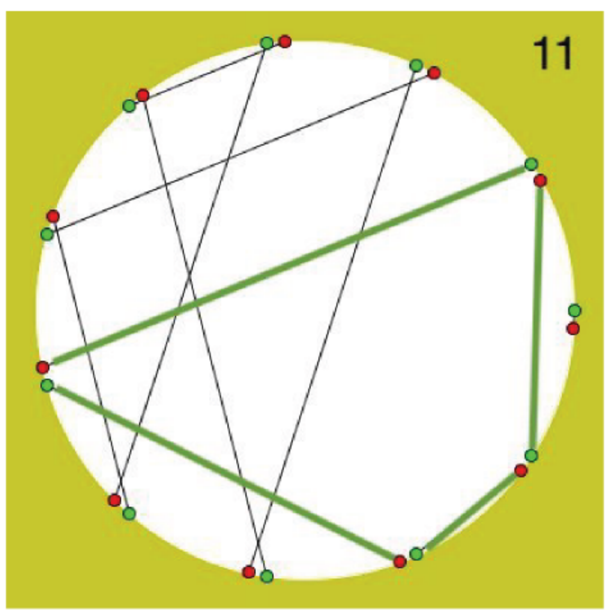
Cliques. I wondered whether the chair swapping would be one long chain - a student takes a chair, the student who was in that chair takes different chair, and so on. (plus the one n→n looper). BUT! I noticed 'cliques' in some of the solution sequences. A clique is a distinct subset of chairs involved in the swapping.

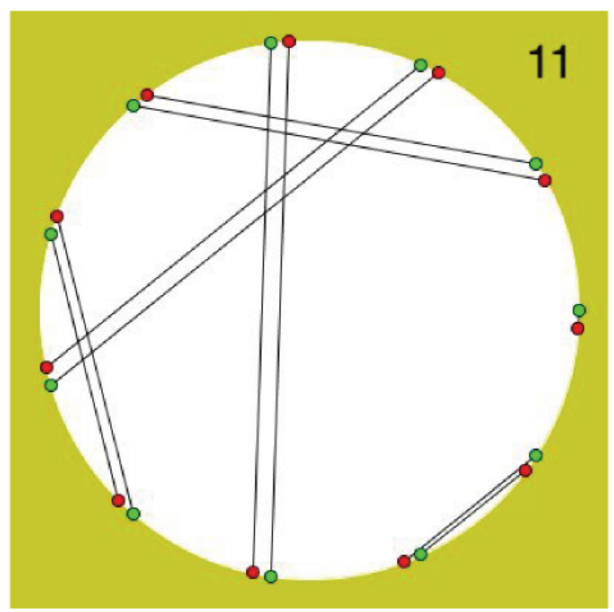
Most solutions are not one long cycle, but contain two or more cliques. See a complete analysis on the website.

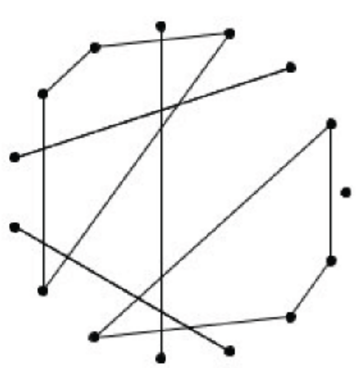
The black & white figure has two 4-cycles, three 2-cycles, and 1 loop. (8+6+1=15)

This turns out to be sequence A003111 on [OEIS], Number of complete mappings of the cyclic group Z_{2n+1}. Thanks to Ian Duff, UK, for pointing this out!

Figures

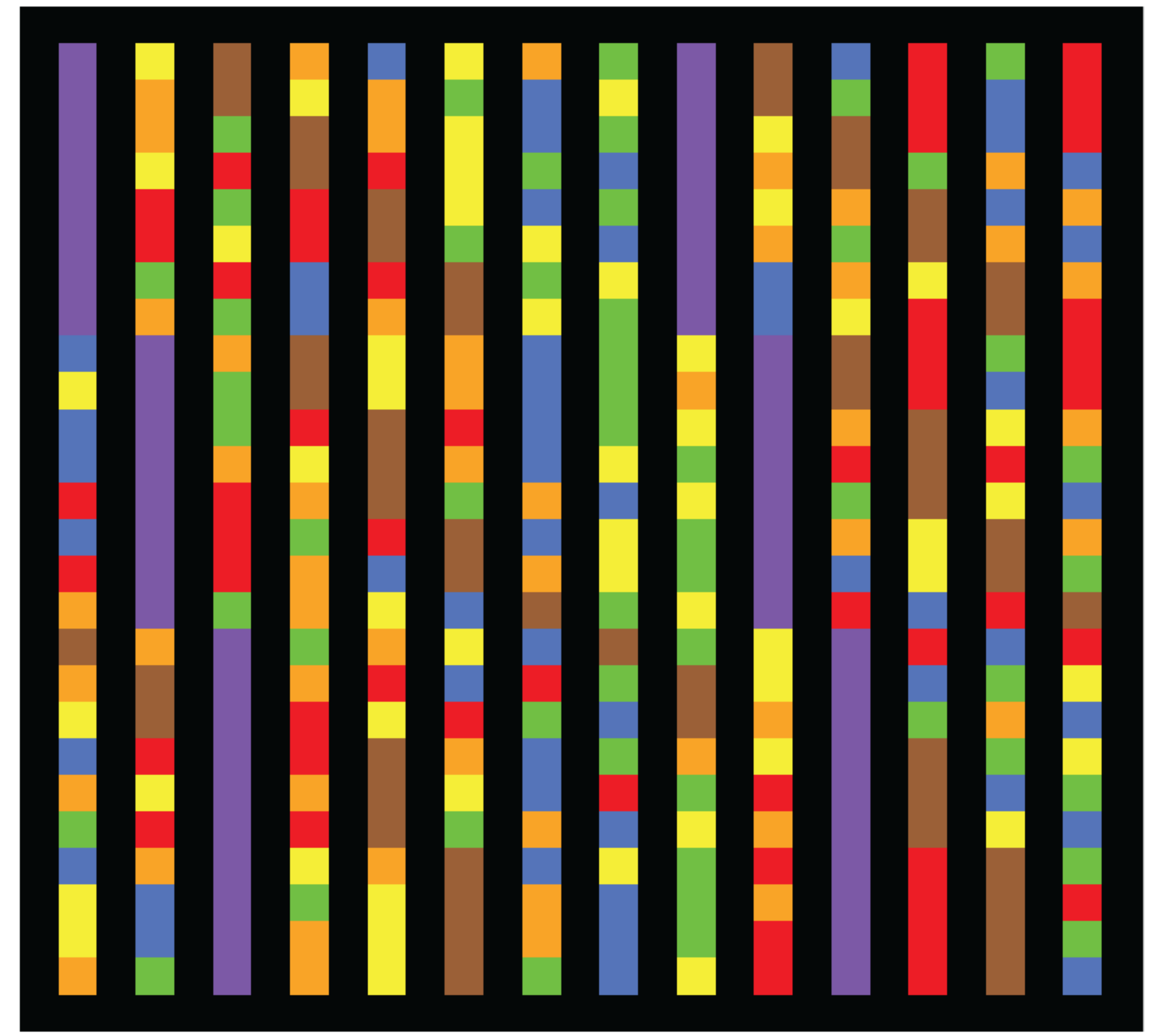






Langford Quilts

A Langford Quilt results when we place solutions together with no space between rows, and use black to separate the colums. This allows the colors to run together serendipitously between rows. This nearly square quilt is made using the 26 solutions for 7 pairs. The quilt has 26 rows, 14+13=27 columns, and a black border.



Exercises

1. Planar connections between pairs in this solution: [4 5 6 7 8 4 1 5 1 6 3 7 2 8 3 2] require five looping joins. Can you find them?

2. How many Colombian Variants are Planar, or End Run Planar? Any? All? Discuss.

3. Prove that Tanton's Chairs can only involve an even number of chairs and students.

References
See http://dialectrix.com/langford.html for all references.

Towers and Dragons: An Unexpected Connection

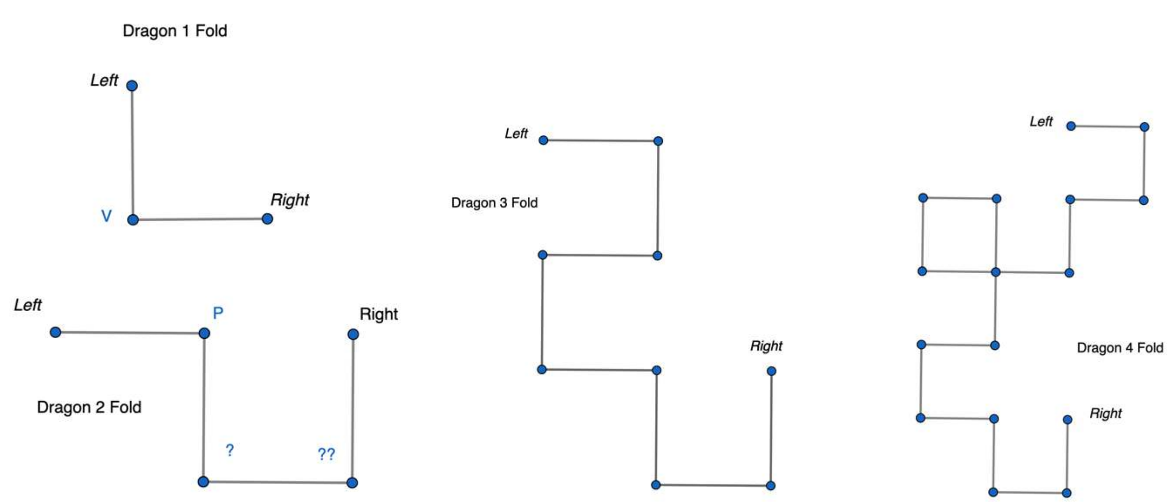
Douglas O'Roark

Introduction
The Tower of Hanoi is a puzzle so well known that it hardly needs an introduction. This paper connects efficient solutions to the Tower of Hanoi and the creases formed when a strip of paper is repeatedly folded in one direction, known as the Dragon Fold. What I've written here is a concise explanation of my "Towers and Dragons" lesson plan that was the winning submission for the 2020 Rosenthal Prize; for a fuller treatment see momath.org/rosenthal-prize/.

The Tower of Hanoi
This puzzle is so well-known that I'll give a very concise description here. N discs are labeled 1, 2, 3,….n, which are decreasing in size as n increases. The discs are stacked on one of three pegs, with the discs decreasing in size as you ascend. The goal of the puzzle is to move the entire stack from one peg to another. The constraints are that you can only move one disc at a time, and that smaller discs can be stacked on top of larger discs but not vice-versa.

The minimum number of moves required to complete the puzzle is $2^n - 1$, which follows by a simple inductive argument. Moreover, the sequence of moves to achieve this minimum follows a symmetric/fractal like pattern. For example, with 4 discs, the sequence of moves is 434243414342434.

The Dragon Fold
Take a strip of paper and fold the left edge to the right edge. Repeat this n times, always folding from left to right. This is a dragon fold, so called because when we unfold the strip the result is a dragon curve:



The first connection to the Tower of Hanoi is obvious: An nth-stage Dragon Fold has $2^n - 1$ creases.

But the connection to Hanoi is deeper still. Suppose we label a crease formed by the first fold 1, then the creases formed by the second fold 2, etc. Read left to right, at the 4th iteration the creases are numbered: 434243414342434! So we can use the creases of the Dragon Fold to solve the Tower of Hanoi puzzle.

An extension: Peaks and Valleys
Another way to label the creases in a Dragon Fold are to indicate a 'V' whenever that crease is a Valley Fold, and a 'P' when the result is a Peak. The first iteration of the Dragon Fold gives the sequence V; the second gives PVV (in the figure above, ? and ?? are both Vs). The 3rd and 4th iterations give:
PPV V PVV
PPVPPVV V PPVVPVV

These, of course, are not symmetric in the same way as the sequences of numbers given earlier. How can we generate each successive iteration of sequences?

First, note that the middle symbol in any iteration is V.
Next, notice how the right side of the sequence for a given iteration is simply the previous iteration.
To derive the left side of the new iteration, visualize unfolding the dragon. What happens to all of the peaks and valleys as we unfold from right to left?!

Tetradecahedron as Palimpsest of the Monododecahedral 1-Parameter Family of the Polymorphic Elastegrity

Subtitle: **Making Paper Bubbles**

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* Received RI NASA summer stipend to work on this project.

Introduction: Origins of the structure from two Bauhaus basic design exercises

In this article, we review and expand upon earlier G4G publications [1, 2] of the polymorphic elastegrity structure that was discovered through paper folding and weaving. Two basic design exercises at the Yale Architecture School led to the discovery of a "paper diamond".

(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 1: 1971 exercise (a) Students were told that design starts by finding simple rules. Anni Albers of the Bauhaus likened it to knitting where the simple rules are to place the yarn over or under a needle. (b) Interest and complexity result from using simple rules. (c) & (d) The simple rules given to students were the only two ways that exist to close pack spheres of equal diameters, as represented with applicator sticks arranged in octahedral-tetrahedral lattices. The vertices represent the centers of spheres that can only be close-packed as (c) A-B, repeating every second layer, and (d) A-B-C, repeating every third layer. Students were also told that 100% of the periodic table crystals are homologous to one or the other close packings, and were instructed to use these two ways of close packing spheres to create interest. (e) A helix with grooves to grow branch helices as seen in the digital recreation of the 1971 finding, resulted from the mechanical repetition of A-B, A-B-C, A-B, and so on. **1972 exercise** (f) Louis Kahn, the famous architect, said to brick, "What do you want, brick?" Brick says to you, "I like an arch." [3] Students were told to allow material to dictate form, in the spirit of Joseph Alpers material exercises;[4] (g) A diagonal crease on a square piece of paper became surprisingly stable and raised the question of what would happen if a second diagonal was added; (h) A second crease created a pyramid. It was recognized as an octahedral fragment due to the familiarity with octahedra gained with the 1971 exercise. It made one wonder, could several paper pyramids make a whole octahedron, and how many? (i) Experimenting with paper showed "paper liked an octahedron" as six crosses of triangles could be assembled into a stable octahedron, by placing two triangles of one axis over, and two triangles of the other axis under adjacent crosses. The resulting octahedron was named a "paper diamond" because it was hard, the number six suggested carbon six, and uncut diamonds come as octahedra.

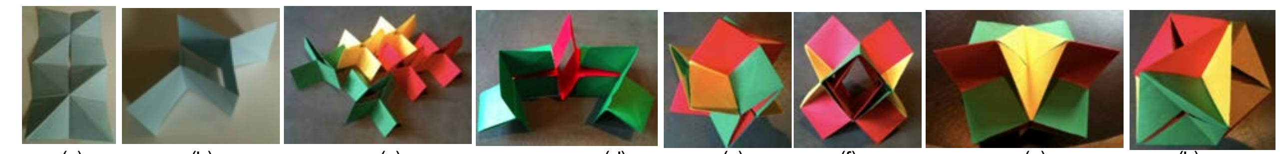
Others independently invented what is named paper diamond here, and called it various names.[5] Diamonds, though, are crystals and crystals grow. Having named it paper diamond, a quest started how to grow paper crystals. Two ways of paper crystal growing were found, one with face connectors fig 2(a), (b), (c) and another with edge connectors fig. 2(e), (f), (g), (h), (i).

(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 2 Face connectors (a) Create a four-cube-corner unit by folding four of the eight octahedral faces; (b) Six four-cube-corner units; (c) Units assembled by inserting a cube into the missing corner creating a strong bond to grow a crystal; (d) Actual diamond under an electron microscope resembles the paper analog of the crystal; **edge connectors** (e) A two-square rectangle with diagonals and crosses creased through the centers of the squares is the element used to create a pyramid with insertable wings; (f) Make a square on the diagonal; (g) Fold the square into a pyramid with wings; (h) Wings inserted in each other to grow the crystal; (i). The pyramids are woven into paper crystals with edge connectors growing the crystal with malleable connections.

A failed experiment leads to the discovery of the Polymorphic Elastegrity

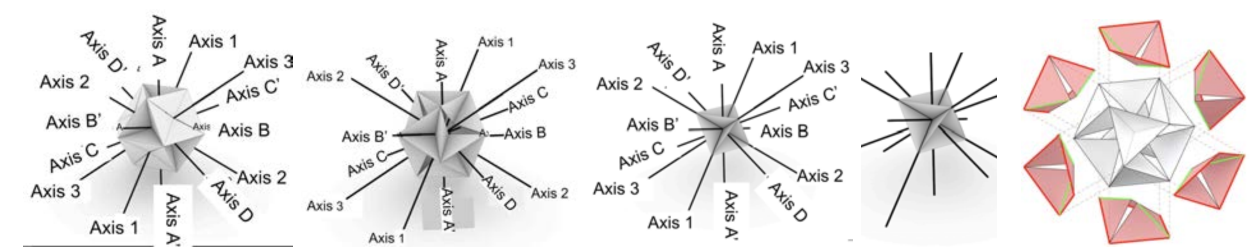
Experimenting to simplify the assembly and make sturdier paper crystals led to further explorations. Creating edge connectors 3(i) as we saw, required a two-square rectangle where each square had a cross creased and two diagonals through their centers. A slit was torn between the two centers of the squares fig 3(a), attempting to discover an improved way of linking crystal units.



(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 3 Discovery of the Polymorphic Elastegrity (a) A slit is torn between the centers of the pre-creased squares; (b) The rectangle is folded axially in half and squeezed to create a cross of eight little squares; (c) One axis of the cross has four squares with a closed ridge on top and open sides. The other axis has four little squares, open on top forming a slit and closed sides; (c) Six units; (d) Weave units placing the side of the cross with the open slit over a side with a ridge on top; (e) Weaving the six crosses of little squares over and under as in step (d), results is three large intersecting squares that do not stay tight together. The slits remain gaping open. It was discarded as a failed experiment. Having forgotten this failed experiment, this flaccid structure was woven again a few months later. (f) Attempting to salvage time spent creating it, the slits were opened; (g) The little squares with the slit on top were folded in half on their creased diagonals into two right triangles hinged along a leg. (h) When all six slits are opened and inverted, twelve elastic hinge systems stabilize the entire structure into an icosahedron. Each hinge system consists of the two right triangles hinged along a leg, with their hypotenuse elastically hinged to a tetrahedron. Each tetrahedron is elastically supported by three hinge systems that link it to three tetrahedra that rotate with opposite chirality.

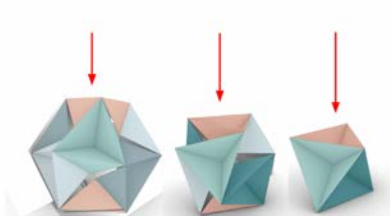
The resulting structure fig. 3(h) has four pairs of tetrahedra along four axes AA', BB', CC', and DD' levitating on six pairs of elastic hinge systems. Each hinge system consists of two right triangles hinged to each other along a leg (shown in green) and along their hypotenuses to two tetrahedra (shown in red) fig. 4(e). Each pair of hinge systems surrounds a gate with four free legs fig. 4(d) that open and close around three orthogonal axes 1, 2, and 3 fig. 4 (a), (b), (c).

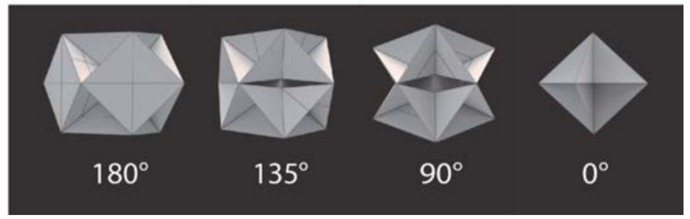


(a) (b) (c) (d) (e)

Fig. 4 Polymorphic Elastegrity (a), (b), (c) Four tetrahedral axes and three orthogonal gate axes that do not move . (a) Expanded into a cuboctahedron, leg hinge dihedral angles 180º, hypotenuse hinge 70.52º; (b) Regular icosahedron, leg hinge dihedral angles 90º, and hypotenuse hinge dihedral angles 28.72º; (c) Contracted into an octahedron, all thirty-six dihedral angles 0º; (d) Seven non-moving axes all motion is in relation to them; (e) Six gates that open and close around the three orthogonal axes. .

When a force is applied along any of the four tetrahedral axes fig. 5, it actuates all thirty-six hinges simultaneously. Twelve leg hinges (red), and twenty-four hypotenuse hinges (green) fig. 4(e), open and close symmetrically and in sync around the three axes fig. 4(c & d). The gates close as the dihedral angles of their leg hinges expand cooperatively to 180º, the hypotenuse hinge angles expand to 70.52º, and the structure turns into a cuboctahedron with closed gates fig. 6 (a). The six gates open in sync in response to compression along any of the tetrahedral axes. When the leg hinge dihedral angles reach 90º the twelve vertices outline a regular icosahedron fig. 6(c). The gates reach their maximum opening at dihedral angles 77.18..º.[6] The gates close again as dihedral angles contract to 0º and turn the structure into an octahedron fig. 6. The tetrahedra rotate in sync as they slide along the tetrahedral axes towards or away from the center, and gates close and open and close again. Four tetrahedra rotate chirally and four tetrahedra located diametrically opposite rotate anti-chirally.





(a) (b) (c) (d)

Fig. 5 A force is applied on one of the four tetrahedral axes expand

Fig. 6 Gates open and close as the twelve leg hinge dihedral angles to 180° cuboctahedron, 90° icosahedron, and contract to 0° octahedron

The asymmetrical tetrahedra form a resilient structure that keeps its shape in elastic equilibrium fig 3(h). When a force deforming is removed, it springs back to its original shape. This is the reason that it was named elastegrity by analogy to tensegrity, which also maintains the integrity of the shape through pre-tension by springing back when a deforming force is removed.

This structure was previously reported at G4G under different names. In 2016 for G4G**12** it yielded a mono-dodecahedron, that is a polyhedron with **twelve** congruent, but not necessarily regular faces. In 2018 appropriately for G4G**13,** its **thirteen** axes were reported. At that time it was still known as chiral icosahedral hinge elastegrity. An editor renamed it Pavlides Elastegrity in 2020 simplifying the name and arguing that structures invented by architects such as the Hoberman Sphere, and the Rubik's Cube are named after the architect who invented them. And since the editor, Elidir King, was classically trained, he also pointed out that Archimedes, inventor of the Archimedes screw, was an architect, the naval architect of Syracusia the largest boat ever constructed in antiquity, as well as an engineer and a mathematician, as Archimedes is more commonly known.

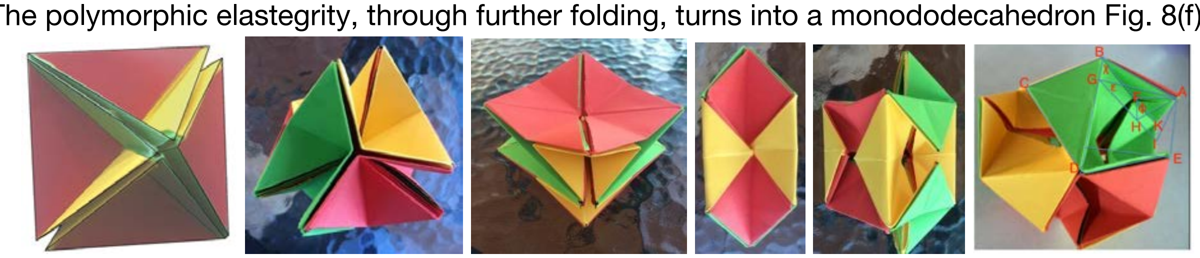
(a) (b) (c) (d)

Fig. 7 Architects who invented structures: (a) Renamed in 2020 Pavlides Elastegrity, (b) Hoberman Sphere, (c) Rubic's cube (d) Archimedes Screw, Architect of the greatest vessel in the antiquity cruise ship, battleship, and freight ship all in one.

However, in 2022, the structure was renamed Polymorphic Elastegrity, due to its shape-shifting properties. First, it contracts into an octahedron and expands into a cuboctahedron, as we saw above fig. 6. With further folding, it can flatten into a multiply covered square and morph into shapes with the vertices of each of the Platonic shapes as presented at G4G12.[1]

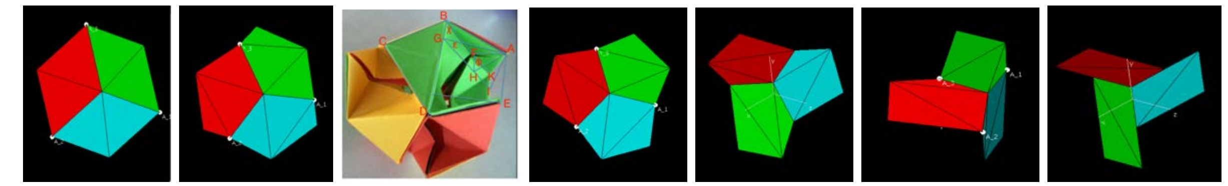
The monododecahedral path

The polymorphic elastegrity, through further folding, turns into a monododecahedron Fig. 8(f),



(a) (b) (c) (d) (e) (f)

Fig. 8 Shape shifting through further folding. (a) Eight tetrahedra contracted into an octahedron; b) The eight rigid tetrahedra are crushed with their right triangle faces bisected through creasing, creating eight groups of triradiational triangles. (c) The triradiational triangles open as shown into flat squares subdivided into four little squares through the creases; (d) Fold the diagonals of the little squares into triangular flaps to cover the slits; (e) The twelve folded flaps create eight pinwheels around the collapsed centers of the tetrahedral equilateral faces, raising the structure into a cube; (f) By lifting the flaps from the face of the cube at a dihedral angle ϕ and angle ε created by AG ⊥ BC so that $\phi + \varepsilon = 90° \Rightarrow \sin\phi + \cos\varepsilon = 1$



(a) (b) (c) (d) (e) (f) (g)

Fig. 9 Monododecahedral path - computer animations by Thomas Banchoff **:** (a) Rhombic (degenerate pentagons one side is 0); (b) Pentagons one angle is smaller than 90°, (c) Pentagon has one right angle; (d) Regular dodecahedron; (e) Pentagon one angle is greater than 108°; (f) Pentagons one angle is much greater than 108°; (g) Rectangle (degenerate pentagon has one angle 180°)

Math proof how to construct a monododecahedron on a cube for any dihedral angle ϕ;

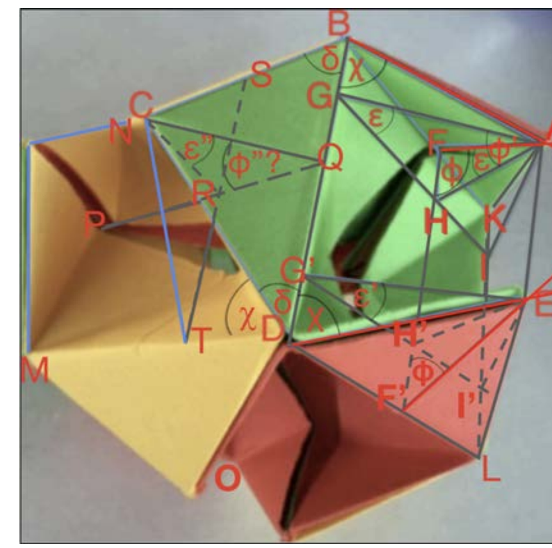


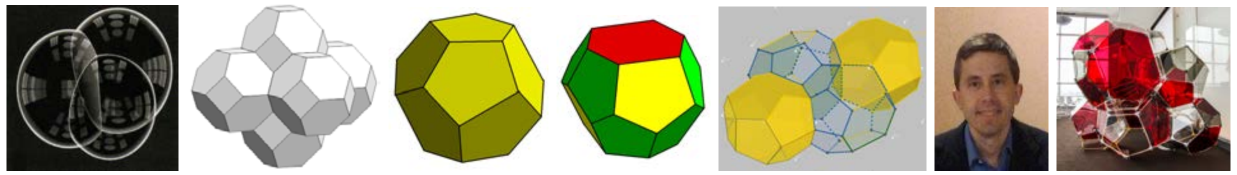
Fig. 10 Monododecahedron

Use fig. 10 to support the proof that for every dihedral ∠ϕ there is a ∠θ= ∠BAK=∠DEL=∠BCD so that ABCDE is flat. What we have is the central cube of the elastegrity in fig. 10, (in what we call the mono-dodecahedron position). What we are looking for is a flap's shape and position such that the vertices of the flaps together with the vertices of the cube form a monododecahedron. The cube is given, the angle of the flap is given. What we need to figure out is where to position the vertices of the flaps in such a way that the resulting figure is a mono- dodecahedron. In particular, ABCDE needs to be flat. (In the physical object the correct positions for the vertices can be realized by further folding the flaps). Draw both planes with dihedral ∠ϕ to face ▫BDLK: (a) plane 1a on through BK, that will contain flap △ABK once A is fixed, and (b) plane 1b through DL that will contain flap △DEL, once E is fixed; Draw plane 2 ⊥ ▫BDLK bisecting it w/ FF'; Intersect plane 1a & 1b w/ plane 2 creating respectively lines L_1 where AF will lie once A is fixed and L_2 where EF' will lie once E is fixed; Draw plane 3 through edge BD w/ dihedral to ▫BDLK ∠ε = 90°-∠ϕ; Intersect plane 3 w/ plane 1a & 1b creating lines L_3 where AB will lie and L_4 where DE will lie once A and E are respectively fixed; Intersect L_1 w/ L_3 to fix point A, and intersect L_2 w/ L_4 to fix point E; Draw AH & EH' ⊥ ▫BDLK & GH||G'H'||BK||DL; △AGH ≅△EG'H' because (a) ∠ε=∠AGH=∠AG'H'; (b) GH = G'H'; (c) ∠AHG = ∠EH'G' = 90°; △AFH ≅ EF'H' because (a) ∠ϕ = ∠AGH = ∠AG'H; (b) AH = EH'; (c) ∠AHF = EH'F' = 90° => AF = EF' => △ABK ≅ △DEL because they are isosceles w/ equal height & base; Extend plane 3 and draw △CBD ≅△ABK ≅ △DEL**;** Given that the dihedral angle between △CBD and trapezoid ABDE = 180° by construction; the dihedral angle between ▫BDMN & ▫BKLD = 90°; the dihedral angle between trapezoid ABDE & ▫BKLD =∠ε by construction; => dihedral angle between △BCD and ▫BDMN ∠ϕ"=∠ϕ
∴ for ∠BAK=∠DEL=∠BCD=∠θ ABCDE is flat Q.E.D

Introducing the Weaire Phelan approximation of minimum tension surfaces

The Polymorphic Elastegrity also yielded through folding the Weaire Phelan mono-dodecahedron fig. 11(c). When arranged in the approximation of the minimum tension surface of bubbles, it leaves tetradecahedra fig. 11(d) as empty space in between fig. 11(e).

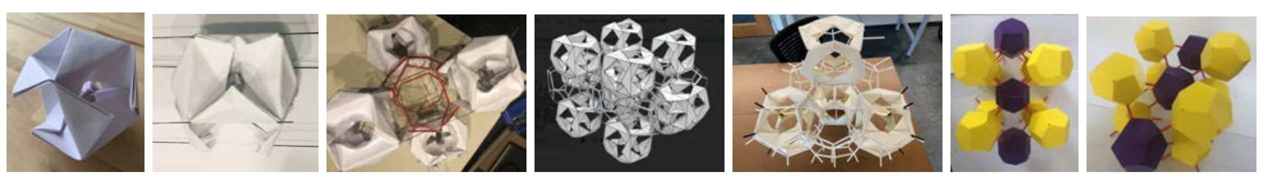
Several authors of this article participated in the 2019 Weaire Phelan workshop at ICERM organized by Glenn Whitney fig. 11(f).[8] We connected edges of three lengths needed to assemble the Weaire Phelan matrix Fig. 11(g), which was an improvement by 0.3% in area[9, 10] over Lord Kelvin's bubble approximation[11] Fig 11(b).



(a) (b) (c) (d) (e) (f) (g)

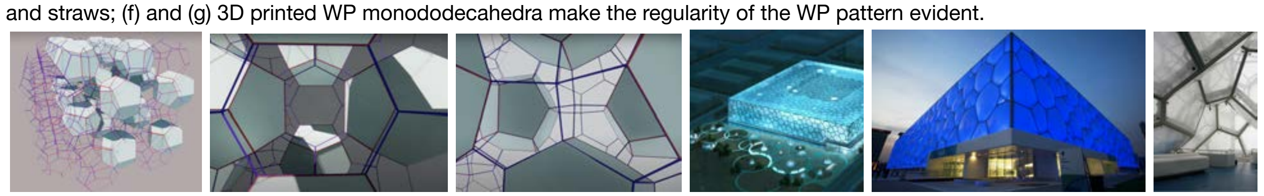
Fig. 11 Minimum surface tension (a) Bubbles; (b) Lord Kelvin's approximation, truncated octahedron 6 squares, 8 hexagons; (c) Weare Phelan 1993 monododecahedron 4 equal sides and one longer, 106.6°, 102.6°, 121.6°, 106.6°, 102.6°; (d) Weaire Phelan tetradecahedron 1887: 4 pentagons congruent to the monododecahedral pentagons; 8 narrow pentagons, 107.02°, 107.02°, 101.54°, 112.21°, 112.21°; 2 hexagons with two parallel sides equal to the monodecahedral pentagon longer side, 126.87°, 116.57°, 116.57°, 126.87°, 116.57°, 116.57°; (e) 2 tetradecahedra & 2 monododecahedra; (f) Glenn Whitney; (g) WP ICERM 2019.

In an epiphany, during the workshop, it became clear that the Weaire Phelan mono-dodecahedron lays along the polymorphic elastegrity path that had already been proven to exist between a rhombic dodecahedron and a cube. It could therefore be obtained through folding paper. Folding the flaps to the exact 121.59° and connecting the paper Weaire Phelan monopododecadra with wire and coffee stirrers would outline the WP tetradecahedra fig 12(a) in the space in between. Appropriate for G4G**14** the tetradecahedron, is a polyhedron with **fourteen** faces, and was literally pulled out of thin air to present and report at the 2022 G4G.



(a) (b) (c) (d) (e) (f) (g)

Fig. 12 (a) The polymorphic elastegrity folded as a monododecahedron, one angle 90°; (b) Paper folded WP monododecahedron 121.6°, 2X106.6°, 2X102.6°; (c) Study model of the tetradecahedron outlined with wire and coffee stirrers between paper folded WP monododecahedra; (d) Sketchup model of paper folded WP; (d) Paper model of WP monodocahedra, 3D printed connectors and straws; (f) and (g) 3D printed WP monododecahedra make the regularity of the WP pattern evident.



(a) (b) (c) (d) (e) (f)

Fig. 13 (a) Digital flythrough WP showing regular staggered rows of monododecahedra; (b) View through a column of tetradecahedra showing alternating orientation of hexagons; (c) View through the narrow tetradecahedral pentagons; (d) Bird's eye view and (e) worm's eye view of the Beijing Olympics Aquatic Center; (f) Interior detail of the Beijing Olympics Aquatic Center.

This article started by citing Anni Albers's admonition to start designing by discovering simple rules and then using them to create complexity and interest. The Beijing Olympic Pool, also known as the "Water Cube", is an example of starting with the highly regular Weaire Phelan structure. The engineer Tristram Carfrae suggested it when the architect Chris Bosse, now of Laboratory of Visionary Architecture (LAVA) proposed a cube of bubbles for the Beijing Olympics Aquatic Center. The architect worked closely with the engineer to choose the plane

to cut through the regular pattern to create interest evoking irregular suds. Given that the structure had physical size, the section needed not to have nodes either just included or just excluded. They cut through the roof with two planes seven meters apart and through the walls three and a half meters apart. Underpinning the whimsical appearance of bubbles of the facade are the simple rules of the Weaire Phelan regular geometry.[12]

Epilogue

Polymorphic Elastegrity was discovered at the intersection of two experiments arising from the Bauhaus approach to design. We saw above how a failed experiment to create an easier assembly and more stable paper crystal in 1982 resulted in this new shape-shifting structure with interesting geometry. Beyond math a matrix of Polymorphic Elastegrity units exhibits -1 Negative Poisson's Ratio along the tetrahedral axes. Unrelated to the better known Poisson's distribution in statistics, Poisson's ratio is the ratio of lateral change over the axial resulting in response to an applied force. For example the ratio of how much a material expands laterally over how much it shortens when squeezed or how much it gets thinner over elongation under tension. Negative is the "perverse" material property as the New York Times called it,[13] when a material gets smaller laterally when squeezed down and wider then pulled. The cooperative retraction of thirty-six elastic hinges suggests engineering applications for energy absorption. Since 2019 the Space Grant Opportunities in NASA STEM -NNX15AI06H has funded students with summer stipends to work on the Polymorphic Elastegrity. Physical and in-silico models (as engineers call animations) brought us closer to engineering applications in shock absorption similar to the tensegrity NASA lander, mechanical analog sensors similar to tensegrity sensors; that could withstand 800º on the Venus surface; and augments the tensegrity conjecture in biology,[14] opening avenues for discovering life on other planets. This work has been reported in the annual reports to NASA: Chris Norcross 2019, Kenneth Mendez 2020, Chelsy Luis 2021.

Fig. 14 Daanish Aleem Qureshi, 2022 recipient of NASA Scholar Summer Stipend working on a matrix of Polymorphic Elastegrities at the G4G14 offsite event, similar to the one he was funded to help with a sphere indentation experiment to measure auxeticity, which is a synonym for exhibiting Negative Poisson's Ratio or NPR, as is it is often abbreviated.

Endnotes

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7 Syracusia, Wikipedia
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Gathering for Gardeners:

A Randomized Approach to Pattern Formation

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Georgia Institute of Technology

As any gardener can attest, ants love to congregate. They relentlessly gather around crumbs, form robust trails, and carry out a variety of simple, coordinated tasks that address the colony's current needs. But how do they coordinate when no single ant is in charge, their communication is limited, and the necessary task may change depending on external conditions known only to a select few? Somehow, ants seem to know exactly when a ripened piece of fruit drops to the ground and, much to the gardener's chagrin, waste no time gathering in a coordinated feeding frenzy. On the other hand, once the food is depleted or they are chased off by an angry gardener, they quickly disperse, all simultaneously executing a new protocol to forage or to run for their lives.

While it is difficult to know exactly what clever ants can accomplish, or how they coordinate so effectively, one can try to model ant-like behaviors with dumbed down *self-organizing particle systems*, where particles, rather than ants, interact via very rudimentary instructions. The question of what can be computed in such a computationally limited, distributed setting is especially compelling because many engineered, physical, and social sciences contain collectives known to self-organize.

Gathering (or *aggregation*) is an illustrative example. How can a system of homogeneous particles, with no global orientation or communication, be made to *aggregate*, forming tight-knit communities, or *disperse*, the inverse action where they spread out and explore? Aggregation and dispersion protocols are found in many natural systems, such as fire ants gathering to form rafts [1] and honey bees communicating foraging patterns by swarming closely within their hives [4]. While each individual ant or bee lacks global knowledge of the collective, it can take cues from its immediate neighbors to achieve global coordination. Similarly, systems of heterogeneous (say, colored) particles can self-organize into either *separated* (or segregated) and *integrated* states, depending on what is most advantageous to the group based on external circumstances. Examples of separation include molecules exhibiting attractive and repulsive forces, strains of bacteria competing for resources while also collaborating towards common goals [6, 7], and social insects acting belligerently or friendly towards other colonies when threats are introduced or removed [5].

A goal for understanding collective behaviors is to find distributed, local algorithms that, when run by each particle independently and concurrently, result in emergent self-organization such as *separation* or *integration* of color classes. In [2] and [3], we presented simple stochastic, distributed algorithms that *provably achieve aggregation/dispersion and separation/integration* by adjusting just a couple of parameters slightly that control each particle's affinity for other nearest neighbors or nearest neighbors of the same color. Adjusting these parameters causes the entire system to undergo a system-wide *phase change*. Thus, each of these collective behaviors can be viewed as *emergent global outcomes of local interactions*, much like phase transitions that turn water into ice or spontaneously magnetize a metal below critical temperatures.

Taking inspiration from these models of biological and physical systems, we now ask what happens if the particles are even more particular about what types of neighbors they prefer. Rather than simply preferring more neighbors, or more like-colored neighbors, what if the particles strongly prefer to have exactly 4 neighbors? What if they prefer 3 red and 3 blue neighbors? Should we expect more phase changes where particles are disordered below some threshold and begin to form long-range organization above some other threshold, much as we see in many particle systems studied in statistical physics?

We explore such questions here with particles that are red or blue and reside on some finite region of the triangular lattice (we add toroidal boundary conditions by identifying left and right sides, as well as top and bottom sides, of a large rhomboidal region so that every vertex on the lattice region has exactly six neighbors). Each vertex is then occupied by a red or blue particle, and particles can swap places, with each trying to find a location where its neighbors have the color ratios they most prefer.

As expected, striking patterns emerge! This is not entirely surprising because we can view red and blue particles as species of ants, and if each prefer six same colored neighbors to five, and five is favored over four, and so forth then we are mimicking the separation algorithm that is known to gather the particles of each color class together. We demonstrate that using other types of *neighbor-aware particles* that favor exactly 3 neighbors sharing their color, or 4, we can get striking patterns of global coordination. What is even more intriguing (to us!) is that the emergent patterns are highly dependent on the density of red particles in the mixture. In addition to long-range order emerging when local affinities are strong enough, we also find remarkable phase changes among emergent patterns as the density is increased, with stripes morphing to polka dots starting locally and spreading over the entire region. Here we demonstrate this behavior with simulations and conjectures.

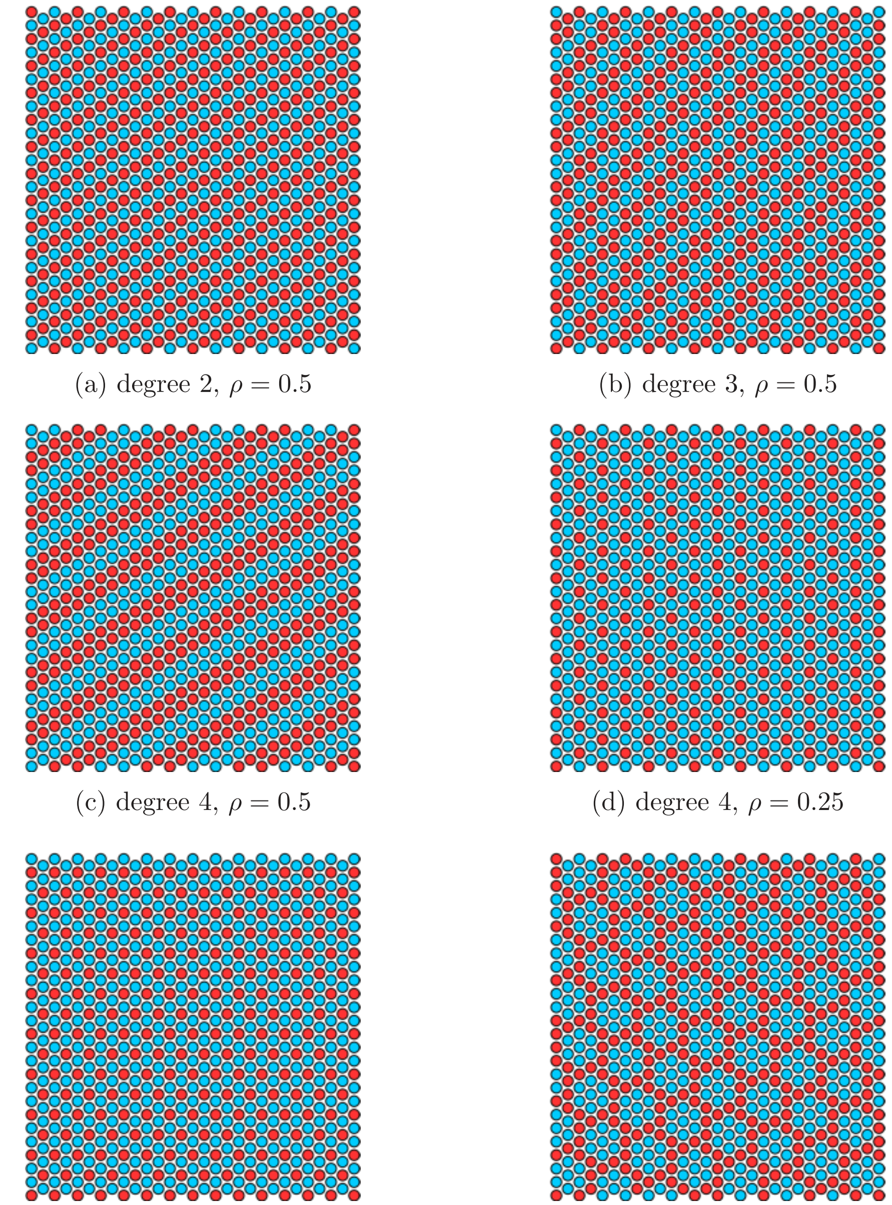
Neighbor-Aware Particles

Let's imagine red and blue colored particles fully occupying all the vertices of a region on the triangular lattice with sites and toroidal (or periodic) boundary conditions. We call this region $G_{\triangle} = (V, E)$, where V are the vertices and E are the edges. We are going to fix the proportion $\rho = N_{red}/N$ of red particles, where $|V| = N = N_{red} + N_{blue}$, the number of particles of each color. The particles know their own color and the colors of each of their immediate six neighbors, and each particle knows its *homophily preference*, i.e., what ratios of like and unlike colors it prefers in its immediate neighborhood.

First, consider what the configurations will look like if we try to maximize the number of particles that achieve their homophily preferences. As an example, consider a region with $\rho = .5$ where all particles have homophily preference of 2, so they want exactly 2 neighbors to have their color and the remaining 4 to have the other color. Figure 1.a shows one way that all vertices can simultaneously achieve this goal. Similarly, when $\rho = .5$ and each particle wants 3 neighbors of each color, then again each particle can satisfy its homophily preference of 3, as shown in Figure 1.b. In Figure 1.c we see how to satisfy every particle's homophily preference when $\rho = 0.5$ and each particle favors 4 neighbors of its own color.

But the striped patterns shown in Figures 1.a, 1.b and 1.c only appear when there are equal numbers of red and blue particles. As we start modifying the density of each, we cannot always make every particle happy and some vertices need to be "sacrificed" to help others. Figures 1.d, 1.e and 1.f show patterns where the maximum number of particles achieving satisfy their homophily preferences. Notice that in Figure 1.d when vertices want 4 like colored neighbors and $\rho = 0.25$, the blue vertices all achieve optimal homophily but none of the red vertices do. We say that this configuration has an *efficiency* ξ of $3/4$ because 75% of the vertices satisfy their homophily preferences.

The last two examples in Figure 2 are for homophily preference 3. In Figure 2.e, $\rho = 1/3$ and the efficiency is $\xi = 2/3$ (meaning a third of the particles are red and the efficiency comes from the blue vertices, which all have the desired three blue neighbors). In Figure 2.f, $\rho = 6/13$ and the efficiency is $\xi = 12/13$ (since all vertices except for the centers of the blue hexagons have the desired homophily preference). The first three examples in this figure all achieve efficiency 1 since all vertices have their optimal homophily values.

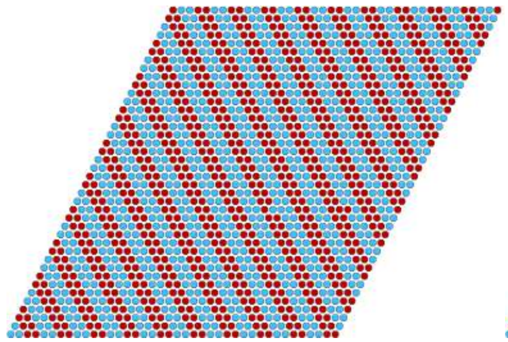


(a) degree 2, $\rho = 0.5$ (b) degree 3, $\rho = 0.5$

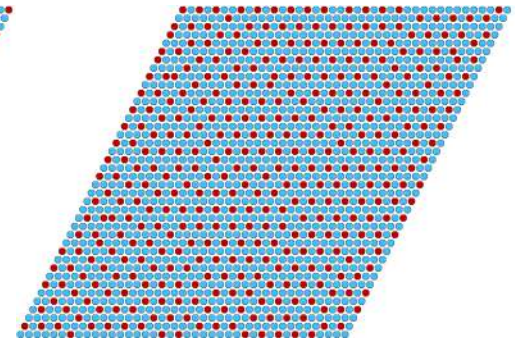
(c) degree 4, $\rho = 0.5$ (d) degree 4, $\rho = 0.25$

(e) degree 3, $\rho = 1/3$ (f) degree 3, $\rho = 6/13$

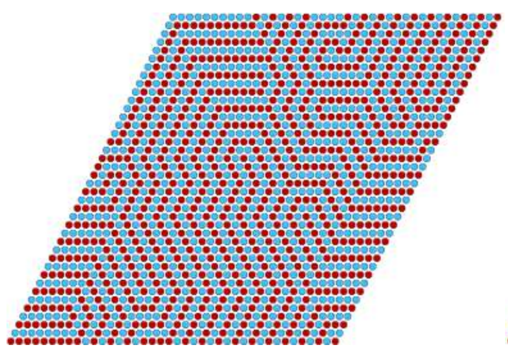
Figure 1: Maximizing vertices with the desired degree at specified densities.



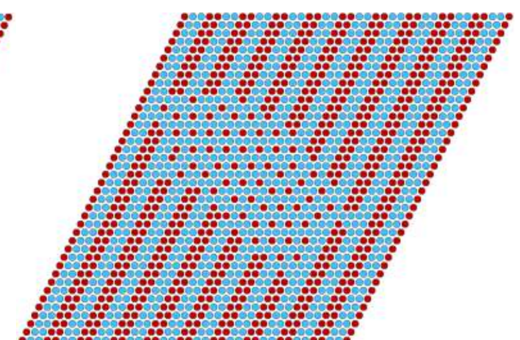
(a) degree 4, $\rho = 0.5$.



(b) degree 4, $\rho = 0.25$.



(c) degree 2, $\rho = 0.5$.



(d) degree 4, $\rho = 0.45$.

Figure 2: Simulations of the probabilistic model with various homophily preferences and densities.

The Probabilistic Setting

A far more compelling situation arises when we define preferences in terms of a probability distribution that, rather than trying to maximize the number of particles that achieve their homophily preferences, just makes such configurations more likely. To do this, we define the weight of any particular configuration to be the product of the individual particles' satisfaction with the colors of their neighbors. More precisely, fix $\lambda_0, \lambda_1, ..., \lambda_6$ and for each i from 0 to 6, let $\lambda_i > 0$ be the relative homophily values that a particle derives when exactly i of its neighbors agree with its own color. Let Ω be the set of valid configurations, i.e., those with ρN red vertices and $N - \rho N$ blue vertices. For any configuration $\sigma \in \Omega$, we define its weight as $wt(\sigma) = \prod_{v \in V} \lambda_{s(v)}$, where $s(v)$ is the number of neighbors w of v such that $\sigma(v) = \sigma(w)$ When we normalize this weight by dividing by the sum of the weights of all possible configurations, we turn this into a probability distribution:

$$\pi(\sigma) = \prod_{v \in V} \lambda_{s(v)}/Z,$$

where

$$Z = \sum_{\tau \in \Omega} \prod_{v \in V} \lambda_{s(v)}.$$

For a homophily preference of 4, for example, we may set $\lambda_4 > 1$ and for all $i \neq 4$, we set $\lambda_i = 1$. Note that as λ_4 gets larger, the distribution starts favoring configurations that have an increasing number of vertices with the desired homophily values.

Figure 2.a shows what happens if we have homophily preference 4 and density $\rho = 0.5$. The orientation of the stripes that emerge can lie in any of three directions and there will always be some defects throughout the pattern that arise randomly. In Figure 2.b we show a homophily preference of 4 and density $\rho = 0.25$. Here we see a grid-like pattern emerging. In Figure 2.c, we have homophily preference 2 with density $\rho = 0.5$. Lines similar to those in Figure 1.a form, but the sporadic degree 3 vertices that arise in the probabilistic setting can cause the lines to curve and wrap around.

In Figure 2.d we see something different. When $\rho = 0.45$, it is not possible to have as many vertices fulfill their homophily preferences as when the density was 0.25 or 0.5. At such intermediate densities, the best one could do is to have part of the region produce a "$\rho = 0.25$" type pattern and part produce a "$\rho = 0.5$" type pattern. This is exactly what emerges when sampling configurations at this intermediate density. Moreover, by nearly minimizing the boundary between these two patterns, we reduce the number of vertices that fail to achieve either nice pattern, and this is also what is observed in Figure 2.d.

We call configurations with patterns that fill the whole region *pure*, such as Figures 2.a and 2.b, and configurations that show multiple patterns simultaneously *mixed*, as in Figure 2.d. Note that since 0.45 is four-fifths the way between 0.25 and 0.5, we expect to see about 4/5 of the region looking like a pure pattern arising from $\rho = 0.25$ and 1/5 looking like the pattern arising from $\rho = 0.5$. In other words, since there is no pure pattern occuring at $\rho = 0.45$, the particle system compromises by optimally mixing the two closest pure patterns in each direction.

Conjectures

Graphically, we can map out what happens for all $\rho \in [0, 1]$ for homophily preference 4 on a diagram. Recall that each "pure pattern" is associated with a

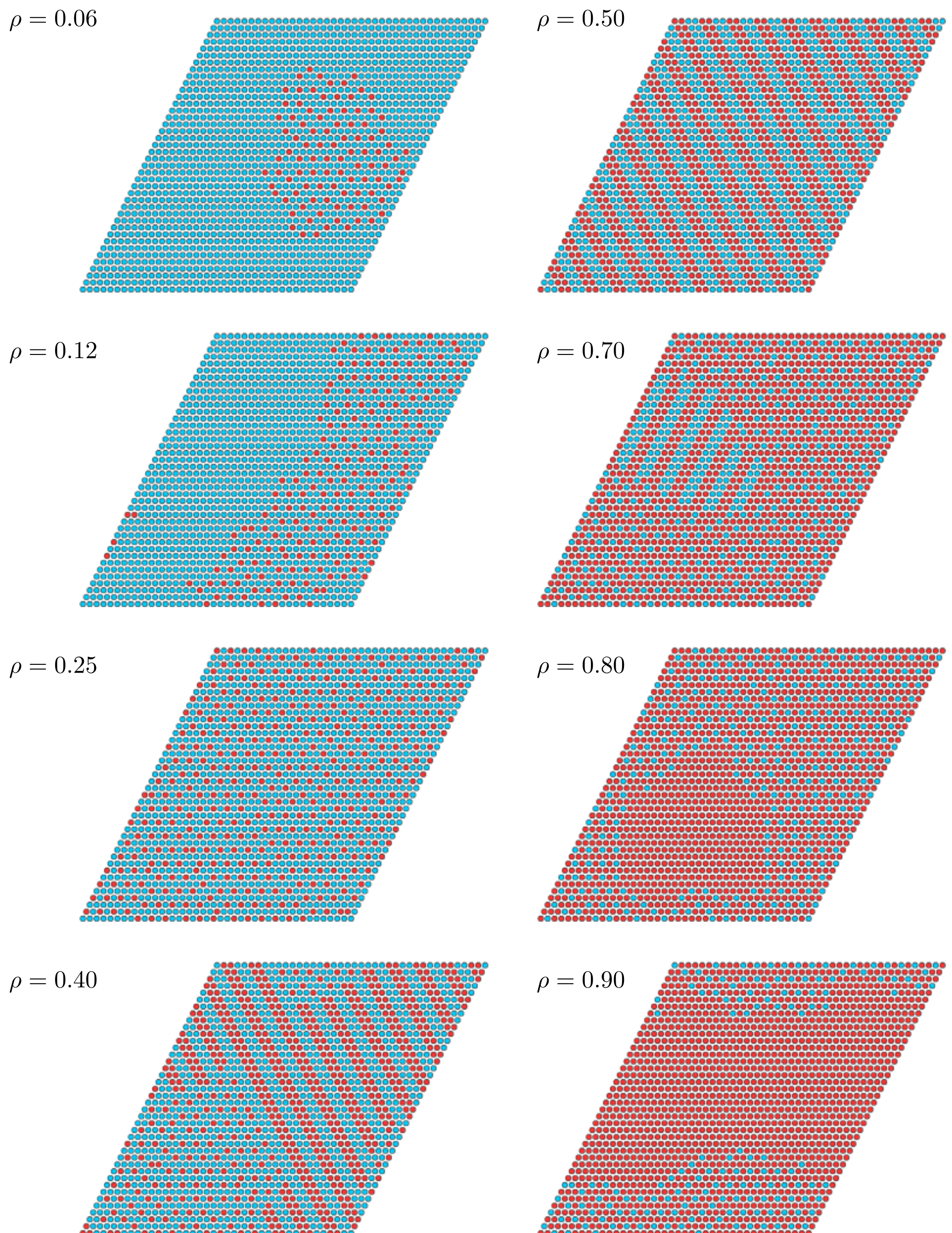


Figure 3: The emergent structures at various densities when we favor degree 4.

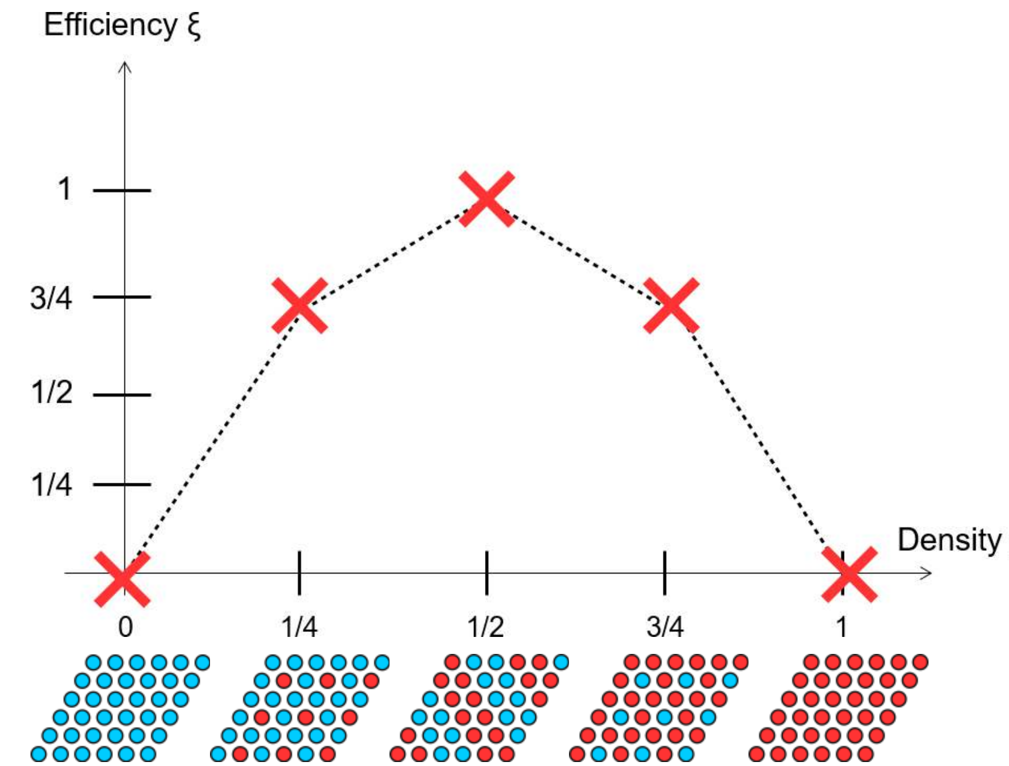


Figure 4: Plot of the obtainable efficiency ξ vs density ρ for homophily preference 4, with the five pure patterns.

specific density ρ, and a specific efficiency. We can plot these pure patterns on a graph of efficiency vs density. There are five such pure patterns for homophily preference 4, which we will first discuss since it represents the more common situation. These are marked as crosses on Figure 4.

Homophily preference 4: For densities in between pure patterns for homophily preference 4, such as $\rho = 0.45$ as shown in Figure 2.d, a mixture of two patterns is obtained. We expect the boundary between these patterns to have length on the order $O(\sqrt{N})$ between these patterns, where N is the number of sites. This means that the efficiency of these non-pure configurations, when averaged over the N sites, is asymptotically equivalent to the interpolated efficiencies of the two pure patterns it lies between. Thus, in between the pure patterns, we draw straight lines representing the optimal efficiency obtainable at each density, as the number of sites go to infinity. The efficiencies on the lines arise by mixing specific proportions of the two adjacent patterns.

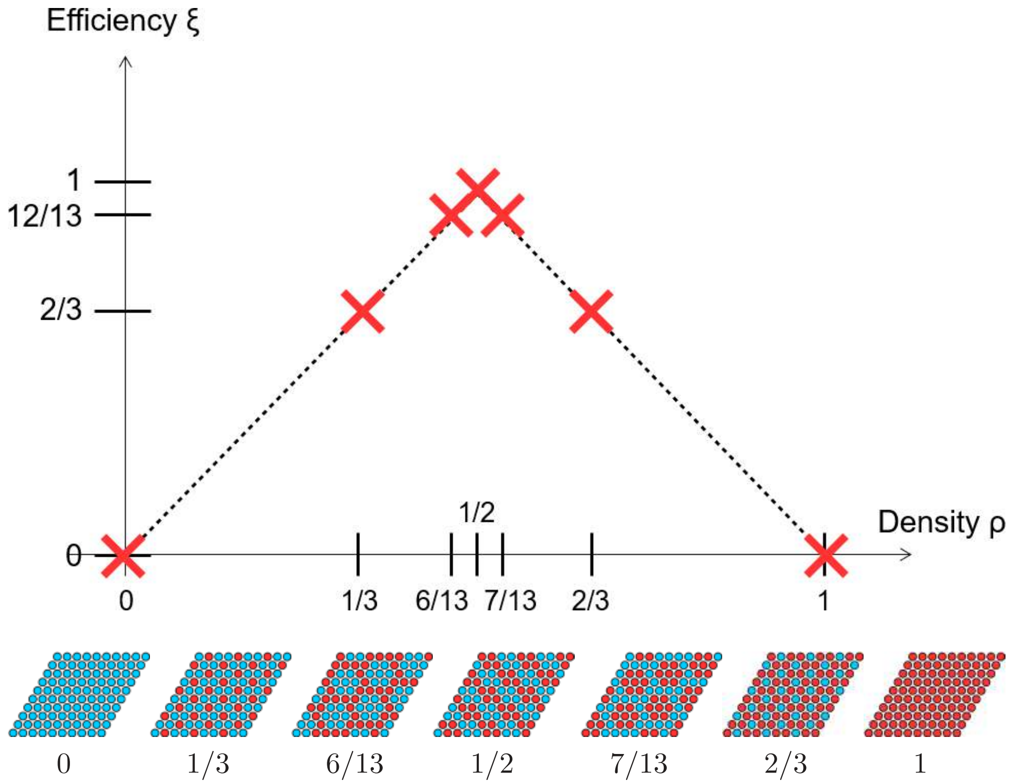


Figure 5: Plot of the obtainable efficiency ξ vs density ρ for homophily preference 3, along with the seven pure patterns corresponding to the red points.

Homophily preference 3: An unusual situation occurs in the case of homophily preference 3, however. The pure patterns correspond to densities 0, 1/3, 6/13, 1/2, 7/13, 2/3 and 1, with efficiencies 0, 2/3, 12/13, 1, 12/13, 2/3 and 0, respectively. Plotting these on a graph of efficiency vs density, we find that the first four points and the last four points are collinear (see Figure 5). The significance of this is that on densities ρ that do not coincide with pure patterns, the interpolated efficiencies can be asymptotically achieved by a variety of mixtures of pure patterns. For example, at red particle density $\rho = 0.25$, one simulation may yield a mixture of the patterns corresponding to efficiencies 0 and 6/13, while another may yield a mixture of the patterns corresponding to efficiencies 1/3 and 1/2. This is in contrast to the homophily preference 4 case, where patterns an intermediate density will always be a mixture of pure patterns immediately to the left and to the right on the plot. The mixed patterns arising from homophily value 3 give rise to far less predictable, but very intriguing, emergent behaviors.

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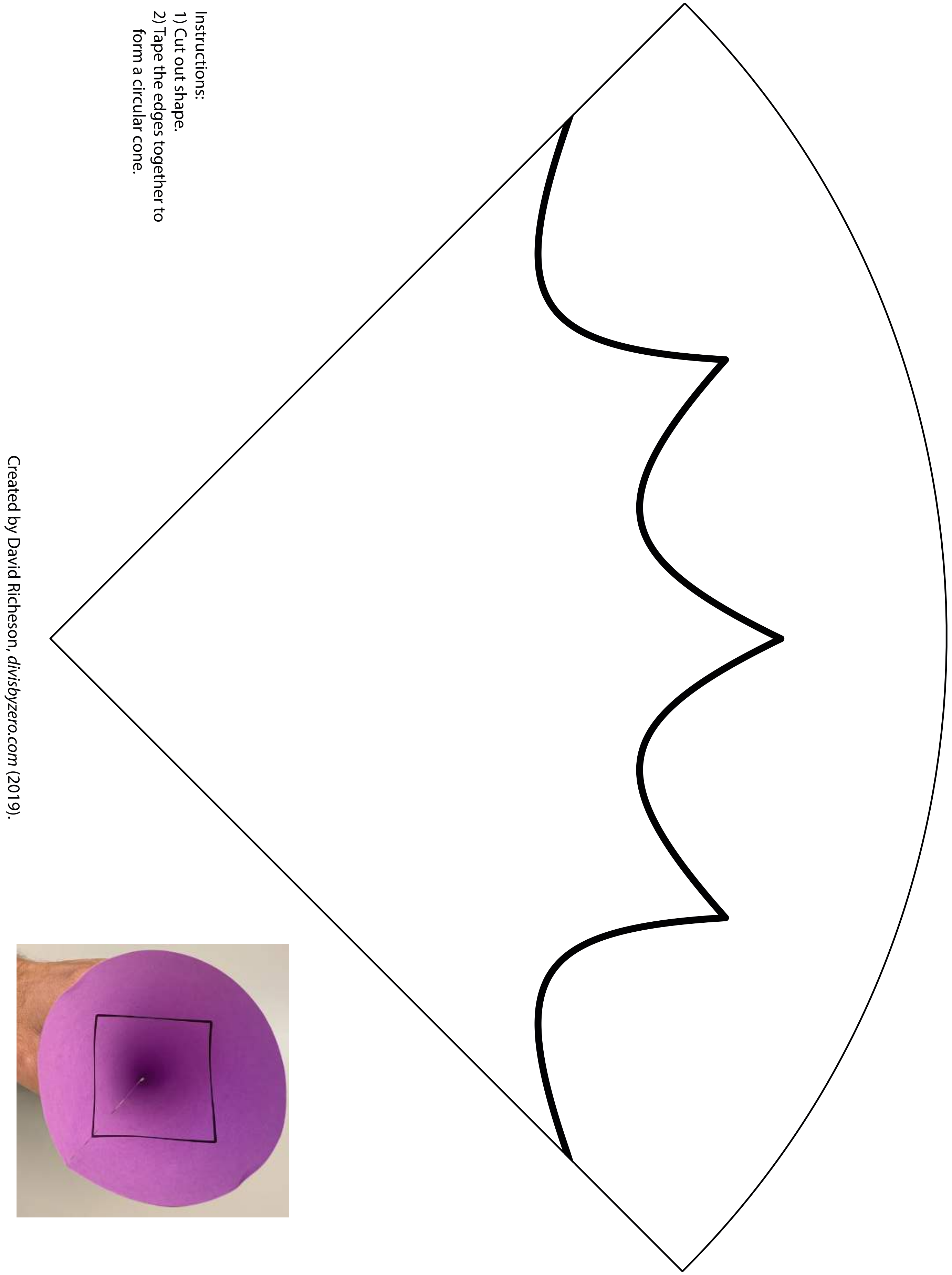
Instructions:
1) Cut out shape.
2) Tape the edges together to form a circular cone.



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Magic and problems from half a millennium ago: The recreational problems of *Tratado da Pratica D'arismetyca* by Gaspar Nycolas, 1519

Jorge Nuno Silva* & Pedro J. Freitas†

The Tratado in its context

By the end of the 15th and early 16th centuries, commercial activity in Europe had become very intense, promoting the transition from a feudal economic system to another in which trade took center stage. In this context, two types of mathematical publications appeared in Portugal: those supporting the navigations and those dealing with problems related to trade.

Regarding the field of Arithmetics, there were about 40 manuals published in Europe between 1472 and 1519. (Almeida 1994, p. 25). This proliferation may have accompanied the establishment of abacus schools, especially in Italy, after the publication of Fibonacci's *Liber Abaci* in 1202, which introduced the use of Indo-Arabic numerals and related algorithms.

In Portugal, three arithmetical treatises were printed, in Portuguese, at the beginning of the 16th century. The first one was *Tratado da Pratica D'arismetyca* by Gaspar Nycolas (Nycolas 1519), which the authors have studied in order to to produce a reedition, to appear soon, and which will be the theme of this paper. The book was first published in 1519 and saw eleven more re-editions (Almeida 1994, p. 82) until the 18th century, the last one in 1716. The author of this text prepared a modern version of Nycolas' book, to appear soon in the Fundação Calouste Gulbenkian's catalogue.[1] The other two treatises were *Prática Darismética* by Rui Mendes in 1540

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[1] https://gulbenkian.pt/en/publications/

(Mendes 1540), and *Tratado da Arte de Arismética* by Bento Fernandes in 1555 (Fernandes 1555).

They all follow a similar structure, starting with the organised presentation of arithmetical procedures, either abstractly or immediately applied to practical cases. It should be noted that all of them use, from the beginning, the Indo-Arabic numerals, completely abandoning the Roman numerals and the operations performed with them, such as the so-called *conta castelhana*, still present in some Spanish textbooks of this period, and which curiously would reappear in a later Portuguese textbook, *Flor Necessária da Arismética* by Afonso Guiral and Pacheco, in 1624. The positional notation, typical of the Indo-Arabic system, greatly facilitates written algorithms which, unlike abacuses, keep the intermediate steps visible at all times, available for inspection.

Focosing now on the arithmetic by Gaspar Nycolas, we see that, after the description of this numerical writing, we move on to the four operations and the algorithms for performing them, which are similar to those we use today. The only algorithm that is considerably different is that of division: the books use the galley division.

After the description of the four operations, we move on to calculation rules, such as the rule of three, which occupies several sections, and which is presented with several variants (which can be considered implementations of the compound rule of three, nowadays abandoned because it can be reduced to iterated use of the of the rule of three). Double false position, an ancient method for solving linear equations, is also used systematically. There are several sections devoted to fractions, and to the extension of the calculation rules to cases where the data are fractional rather than integer. Then we find some sections devoted to practical problems on taxes or bartering. There are sections mainly of numerical problems, a long section on geometry, and another on methods for extracting square and cubic roots. At the end of the book there are several problems on silver alloys.

Contrary to the pedagogical principles to which we are accustomed today, the solution to these problems is mostly presented without explanation — the author begins the resolution with the expression “Do it this way” and describes the method for solving the problem (in many cases it is not immediately clear why the given resolution actually solves the problem). The motivation, clearly, was to mechanise these methods so that the reader could put them into practice expediently in the daily problems of commerce.

Alongside these pragmatic considerations, we find a long collection of problems of a recreational nature. It was already a medieval tradition, that of accompanying the mathematical textbooks by lists of problems intended to develop reasoning and to entertain. Gaspar Nycolas explicitly refers to Luca Pacioli’s *Summa* as a source for these problems, but some of them come from older traditions, both medieval European and classical Greek. Some of these problems can be solved using the methods presented earlier in the book, but others need an *ad hoc* reasoning, or, as the author says, can only be solved “by fantasy”, that is, by thinking of a resolution specifically for the given problem.

For the sources of problems, we searched for previous occurrences in the literature, namely in the most well-known collections, namely: Metrodorus’ *Greek Anthology* (Paton 1980), Alcuinus’ *Propositiones ad Acuendos Juvenes* (Hadley and Singmaster 1992), Fibonacci’s *Liber Abaci* (Sigler 2002), Treviso’s *Arithmetic* (Swetz and Smith 1987), Pamiers’ *Arithmetic* (Sesiano 2018), Chuquet’s *Le Treviso* (Chuquet 1881, Chuquet and Marre 1881), *Summa* (Pacioli 1494) and *De viribus quantitatis* (Pacioli ca. 1509, Hirth 2015) by Luca Pacioli and *Conpusicion de la arte de la arismetica y Juntamente de geometria* by Juan Ortega (Ortega 1512.

Recreational mathematics, often immersed in works of another nature, has often been little noticed, even decried, by scholars. However, it is increasingly becoming unavoidable in the historical approach. Its roots are thousands of years old and it can no longer be denied that recreational motivation is present in a relevant part of the evolution of mathematics (Singmaster 2017). Of the three arithmetic books we mentioned, it is the one by Gaspar Nycolas that devotes the most time to topics of recreational mathematics, about a third of the book.

A few selected problems

Sum and product equal

> Give me a number that is the same, either added and multiplied. You may know that there is no other whole number except 2, because 2 and 2 is 4 and 2 times 2 is 4. However, let’s exclude this one, and try and find two numbers that give the same result both added and multiplied.
>
> Here’s a general rule for such questions. Take any number, whichever you want, and divide it by one of its parts, whichever you want, and whatever is missing from that part, you will divide it again by that remainder, and whatever comes, these are the numbers demanded.

> For example, take 7, divide by 4 and you get 1 and $\frac{3}{4}$, this is one of the numbers. To know the other one, take the difference between 7 and 4, which is 3. Divide 7 by 3 and you get 2 and a third, this will be the other number. The sum of these numbers is 1 and $\frac{3}{4}$ plus 2 and a third, which is 4 and $\frac{1}{12}$. If you now multiply one and $\frac{3}{4}$ by 2 and $\frac{1}{3}$ you get the same 4 and $\frac{1}{12}$.

This problem starts with the following question: find a number x such that $2x = x^2$. There are only two numbers with this property, 0 and 2. The text suggests that this question would be something of a riddle, as it adds that when asked, one should immediately exclude the number 2. Next, the problem is slightly modified: two numbers are now asked such that their sum is equal to the product. The author then gives a rule for finding such numbers: write a number a as the sum of two parts $a = b + c$ and take the numbers a/b and a/c. And indeed, their sum and product is a^2/bc. The author gives no indication about the origin of this method.

Generating squares

> Give me a number that, if you take away 11 from it, it becomes a square, and if you put 10 on it, it also becomes a square.
>
> This is the method: join these quantities that you want to take and put, and from that sum always take one. Divide in the middle what remains, and that half always multiply in itself. To this multiplication you will add that amount that you want to take it out, and after all this, you'll get be the number you were asked for.
>
> In this example: add 10 and 11, you get 21. Take 1 and 20 remain, take half of that, which is 10. These I say multiply in themselves, and you will make 100. To these you must add the amount you want to take out, and you get 111. And this is the number that if you take 11 from it, you get a hundred, which is a square, whose root is 10, and if you add 10 you will also get a square number, which is 121 whose root is 11.

We are looking for a number x such that

$$\begin{cases} x - 11 &= \square \\ x + 10 &= \square \end{cases}$$

where $\square$ represents a generic perfect square. The procedure given by the author leads to the solution $x = 111$ ($111 - 11 = 10^2$, $111 + 10 = 11^2$).

In general, given two integers a and b, with odd sum s, one is asked to calculate

$$\left(\frac{s-1}{2}\right)^2 + a$$

Naturally, subtracting a, the result is a perfect square. If we add b to it, we get

$$\left(\frac{s-1}{2}\right)^2 + s = \frac{s^2 + 2s + 1}{4} = \left(\frac{s+1}{2}\right)^2$$

The perfect squares obtained by this method are always consecutive. Other solutions can be obtained, noting that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, and therefore the difference of two squares, possibly non-consecutive, is always the sum of consecutive odd numbers. In this case, we can write $21 = 5+7+9$, so $21 = 1 + \cdots + 9 - (1 + 3) = 25 - 4$, and therefore $4 + 11 = 15$ would be another solution. As 21 cannot otherwise be expressed as the sum of consecutive odd numbers, these are the only solutions on the integers.

A broken weight

> A man had a stone that weighed 40 *arráteis*, it hit the ground and was broken in four pieces. With these 4 pieces he produced any number of *arráteis* as he was asked for, from one to 40. Now I demand how much each of them weighed.
>
> Know that this one has no rule [...] it is made of fantasy.[2] Know that one of the pieces has one *arrátel* another has three and the other has 9, which is the square of three, and the other has 27, whose cubic root is three. But this rule is not general, it is by fantasy. So you will say that from the four numbers 1, 3, 9, and 27 the four pieces were made.

This problem appears already in Fibonacci (Sigler 2002, p. 420), Chuquet (Chuquet and Marre 1881, p. 451, CXLII) and in Pacioli (Pacioli 1494, F97, 34). However, its origin is more remote, Topfke references a Persian occurrence in the XI century (see Tropfke et al. 1980, p. 633).

Presumably the problem concerns the use of a balance scale, with two plates, in each of which you can place either merchandise or weights. Thus,

[2] Here, "fantasy" must be understood as reasoning without application of any standardised procedure.

we look for four positive integers, x_1, x_2, x_3, x_4, whose sum is 40 and such that any natural n, $1 \leqslant n \leqslant 40$, can be expressed as a linear combination

$$n = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$$

where the coefficients α_i can take the values $-1, 0, 1$. The author presents the solution, justifying it with "fantasy". Chuquet states that the sequence of weights starts with 1 and then each weight is one unit plus twice the sum of those that precede it. Pacioli starts with 1 and successively multiplies by 3, which is equivalent. Fibonacci mentions both procedures.

There is an interesting detail in the way the author poses the problem: in no other source the four weights are the result of a larger object being broken. This particular setup tells us that probably Nycolas had access to another source, possibly not so well-known as the ones we refer to.

Bags

> There are two bags, one holds 8 *alqueires* of wheat and the other holds two. Now, I unsew them and make a bag that's as tall as they were before. I ask how many *alqueires* the big sack holds.
>
> Do it this way. Combine 2 with 8 and that's 10, save these. Now multiply the bags against each other, meaning 2 times 8, which is 16, take the root which is 4, double it, and it is 8. These 8 together with the 10 that I ordered you to save and are 18, and these many *alqueires* will the big sack hold.

Let's assume that sacks are obtained from two overlapping rectangles of fabric sewn along three sides, like the so-called burlap sacks. It is natural to assume that the volume of each bag is proportional to the square of the base seam. As the height is constant in this problem, we can suppose it to be unitary and, being the measures of the base seams c_1 and c_2, the volumes will be $8 = c_1^2 k$, $2 = c_2^2 k$, for some constant k. Joining the pieces of cloth, we obtain a sack with a seam at the base measuring $c_1 + c_2$. The volume will then be

$$(c_1 + c_2)^2 k = c_1^2 k + 2c_1 c_2 k + c_2^2 k = c_1^2 k + 2\sqrt{c_1^2 k c_2^2 k} + c_2^2 k = 8 + 8 + 2 = 18$$

as in the text. Alternatively, if we consider the sacks as the side surfaces of cylinders, assuming unit height, the problem is reduced to determining the area of the circle whose perimeter is the sum of the perimeters of two circles of areas 8 and 2. The result is, again, 18.

Break 9

> Divide 9 into two parts such that, dividing the greater by the smaller, I get 19. This is the rule: add one to the number that you want to obtain, this will be your divisor, and the dividend is 9. Therefore, divide 9 by 20, and you get 9/20, this is one of the numbers. The other will be $8\frac{11}{20}$ as you can prove.

The system to be solved is $x + y = S$ and $x/y = Q$ (with $S = 9$ and $Q = 19$). From the second equation, we get $x = Qy$, and substituting into the first equation, we get $(Q+1)y = S$, which is the solution shown.

Conclusion

The *Tratado de Prática Darismética* can be seen as a source for understanding the commercial mathematics of the 16th century, crucial for Portugal and relevant for the rest of Europe. In the teaching tradition of practical mathematics, Nycolas' text shines as a sophisticated pedagogical book. Including both pragmatic exercises, solved by application of general rules, together with recreational problems, which entertain with their appeal to imagination and "fantasy", the author offers the students in abacus schools (and, more generally, those training for commercial matters) a glimpse of abstract rigorous thought in a ludic context.

We believe that the *Tratado*, being the first book on mathematics printed in Portugal, contributed decisively to the dissemination of mathematical teaching, now aided by the printing press, which permitted a much faster and secure expansion of the subject. Its popularity can be gauged by the number of editions of the book (up until the 18th century), meaning that it became a source of mathematical learning for several generations of Portuguese merchants, accountants, and likely also people interested in a good problem to wrap their heads around.

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Traveling through the Sierpinski carpet and Menger sponge

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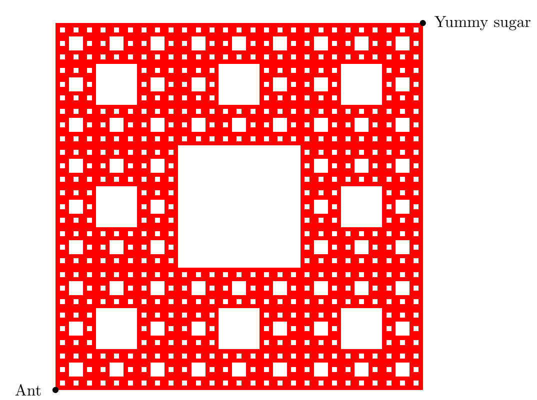
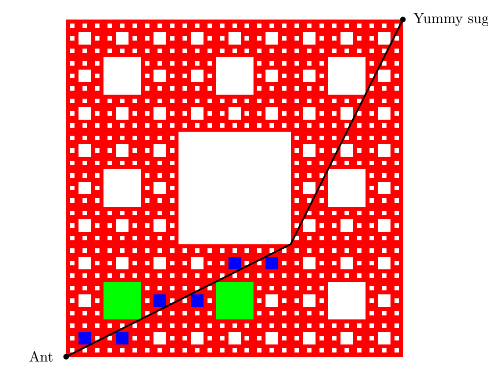
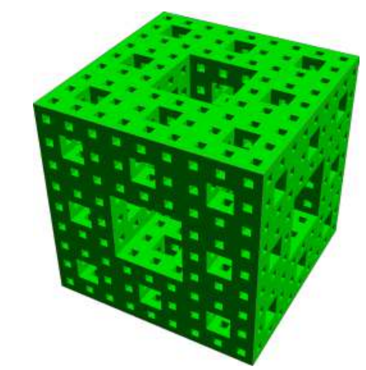
The Sierpinski carpet is an intriguing 2-dimensional fractal. You can construct the carpet by taking a solid square of side length 1, dividing it into nine sub-squares of side length 1/3, and removing the "open" sub-square in the middle, *i.e.* remove all of the points of that middle sub-square except for the points on its boundary edges. For each of the eight remaining sub-squares, repeat the procedure above by dividing it into 9 sub-sub-squares and removing the middle one; and for each of the 64 remaining sub-sub-subsquares repeat the procedure; and so on; keep going; you're not done yet! The limiting object is the Sierpinski carpet. The red figure below is just an approximation of it, after repeating the subdividing and removal procedure only 4 times, but it should be enough to give you a good sense of things.



Problem 325 in the April 2015 issue of *Math Horizons* asked the following question. An ant starts at one corner of the fractal and wishes to travel to the opposite corner while staying on the fractal. It's clear that the ant can do this by traveling a distance of 2, simply by moving along two exterior edges. But can the ant get to the opposite corner by a shorter path? What is the shortest path the ant can take to get from one corner of the Sierpinski carpet to the opposite corner while staying on the fractal and not falling into any open square hole? **Don't go to page 2 until you've tried your best to get a path whose length is shorter than 2!**

In fact, the ant can do much better than a distance of 2 by going through the interior of the carpet. If the only paths through the carpet use line segments parallel to the outer edges of the carpet, the ant wouldn't be able to beat a distance of 2. But there are line segments in the carpet that have slopes of 1/2, 1, 2, and their negatives! The figure on the left below shows one of two shortest possible paths from the lower left corner to the upper right corner, with a total distance of $(2/3)\sqrt{5} \approx 1.49$. Try to use the self-similar structure of the carpet and the positions of the green and blue holes relative to the lower line segment of slope 1/2 to convince yourself that this path is, indeed, contained in the Sierpinski carpet. Also, try to find a line segment in the carpet of slope 1!





A 3-dimensional fractal that is closely related to the Sierpinski carpet is the Menger sponge: it is a cube with Sierpinski carpets on its faces, and with open tunnels bored straight through the cube where there are open square holes in the carpet faces. An approximation of the Menger sponge is shown above in green.

The problem in *Math Horizons* had a second part. Suppose a termite wants to travel from one corner of the Menger sponge to the opposite corner. A path of length 3 can be had by following three exterior edges of the sponge... but can the termite do better? Maybe it's via a path that stays on the outer Sierpinski carpet faces of the sponge, but we also allow the termine to bore through the sponge, always staying in the green material of the sponge and avoiding any removed open tunnels. What's the shortest possible route?

Try to use what you now know about paths in the Sierpinski carpet to help you find a path that stays on the surface of the Menger sponge and has length less than 5/2. Then, for the real challenge: try to find a path that can be bored through the sponge and has length less than 2!

Good luck! If you would like some hints, or some good references on geodesic paths in fractals, just send me an email. As for the shortest possible path in the Menger sponge from one corner to the opposite corner... this problem remains unsolved.

This work is joint with Ethan Berkove.

An Unexpected Appearance
(Or Look Ma, No Rectangles)

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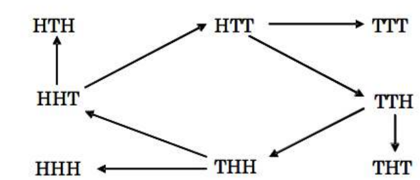
Many papers are a straightforward presentation of the facts,but sometimes there are papers containing a nice surprise. Two of the more exciting types of these gems in mathematics are an object with a counter-intuitive property and the surprise appearance of something from an unrelated topic. In this note we have both of those elements. Our surprise appearance comes from a particular instance in Penney's Game which itself comes from a fun exercise whose base idea is counter-intuitive. This begins with a problem submitted by Walter Penney to the *Journal of Recreational Mathematics* ([6]) in 1969.

> `Although in a sequence of coin flips, any given consecutive set of, say, three flips is equally likely to be one of the eight possible, i.e., HHH, HHT, HTH, HTT, THH, THT, TTH, or TTT, it is rather peculiar that one sequence of three is not necessarily equally likely to appear` *`first`* `as another set of three. This fact can be illustrated by the following game: you and your opponent each ante a penny. Each selects a pattern of three, and the umpire tosses a coin until one of the two patterns appears, awarding the antes to the player who chose that pattern. Your opponent picks HHH; you pick HTH. The odds, you will find, are in your favor. By how much?`

The ensuing game became popular after an appearance in Martin Gardner's *Scientific American* column ([2]).

Penney's Game is a two-player game played via the flipping of a fair coin. Player I picks a sequence of Heads or Tails of length 3 (it can be any agreed-upon length, but in most all literature it is length 3) and makes his choice known. Player II then states her own sequence of length 3. An umpire then tosses the coin until one of the two sequences appears as a consecutive subsequence of the coin flips. The player whose sequence appears first is the winner.

This is one of the members of the collection of *non-transitive games*. Just like Rock-Paper-Scissors regardless of Player I's decision, Player II always has a choice that puts the odds in his/her favor. We show this with the following diagram where the better choice lies at the base of the arrow.



Here are the probabilities associated with the diagram.

Player I's Choice	Player II's Choice	Probability Player II wins
HHH	THH	7/8
HHT	THH	3/4
HTH	HHT	2/3
HTT	HHT	2/3
THH	TTH	2/3
THT	TTH	2/3
TTH	HTT	3/4
TTT	HTT	7/8

Several methods exist to determine the result, including one using of martingales and one due to John H. Conway that uses the binary representations of numbers ([1], [5], [4]).

In this note we focus on one specific situation: Player I has chosen HHT and Player II HTT[1]. This leads us to the surprising appearance.

Assume for our (unfair) coin that p is the probability that the coin land on Tails. Define $x = P(\text{HTT wins})$. We want to find x in terms of p. Now if the opening consists of a string of T's, this has no affect. Similarly, the first H gives neither player an advantage. So assume we begin with $n \geq 0$ T's followed by one H:

$$\text{T}\cdots\ \text{TH}.$$

After the H appears in order for Player I to *not lose* we have either

$$\text{T}\cdots\ \text{THTT, where Player I wins}$$

or

$$\text{T}\cdots\ \text{THTH, where we are in the same situation as T}\cdots\ \text{TH}.$$

[1]We realize that this is not optimal play with a fair coin. The interesting result arose when trying to determine all optimal plays when the coin is not fair.

Thus

$$x = p^2 + p(1-p)x$$

which leads to

$$x = f(p) = \frac{p^2}{1-p+p^2}.$$

Now the question is, "Can this be a fair game?" What should be the probability of landing on Tails in order for $P(\text{HTT wins}) = P(\text{HHT wins}) = 1/2$? Setting $f(p) = 1/2$ and solving for p yields

$$p = \frac{-1+\sqrt{5}}{2}.$$

Here is our surprising appearance! This value of p is $1/\Phi$, where Φ is the *Golden Ratio*. Although there is debate about the aesthetic properties of the Golden Ratio ([3]), this result does show us Φ can appear when least expected.

A more thorough investigation of Penney's Game using weighted coins can be found in [7].

References

[1] Collings, Stanley, "Coin Sequence Probabilities and Paradoxes," *Bulletin of the Institute of Mathematics and its Applications*, Vol. 18, November/December 1982, 227 – 232.

[2] Gardner, Martin, "On the Paradoxical Situations That Arise from Nontransitive Relations," *Scientific American*, October, 1974, 120 – 125.

[3] Markowsky, George, "Misconceptions About the Golden Ratio," The College Math Journal, Vol. 23, No 1 (Jan 1992), 2 – 19.

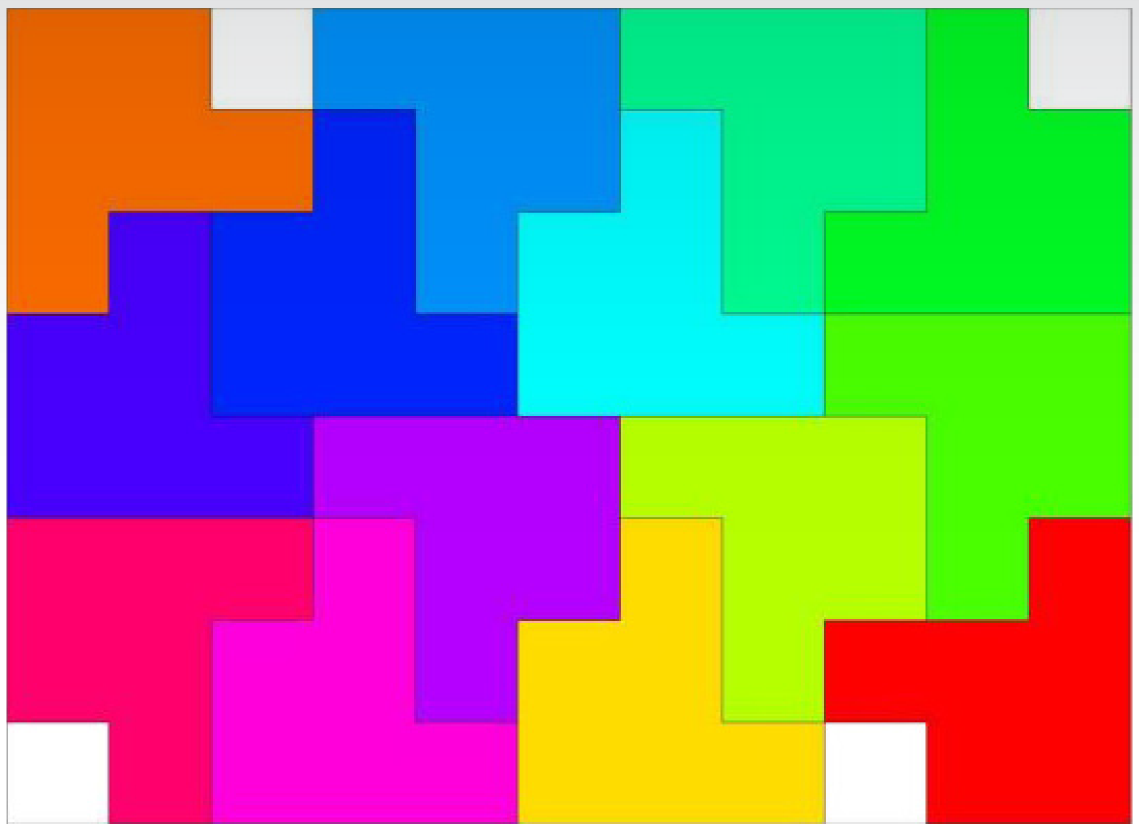
[4] Montgomery, Aaron and Robert W. Vallin, "Penney's Game and Martingales," Proceedings of Recreational Mathematics Colloquium V–G4G (Europe), 2017, 100 – 109.

[5] Nishiyama, Yutaka, "Pattern Matching Probabilities and Paradoxes as a New Variation on Penney's Coin Game," Osaka Keidai Ronshu, Vol. 63 No. 4, November 2012, 269–276.

[6] Penney, Walter, "Penney-Ante," *Journal of Recreational Mathematics*, Vol. 2, 1969, 241.

[7] Vallin, Robert, "Penney's Game with an Unfair Coin," (submitted).

PUZZLES



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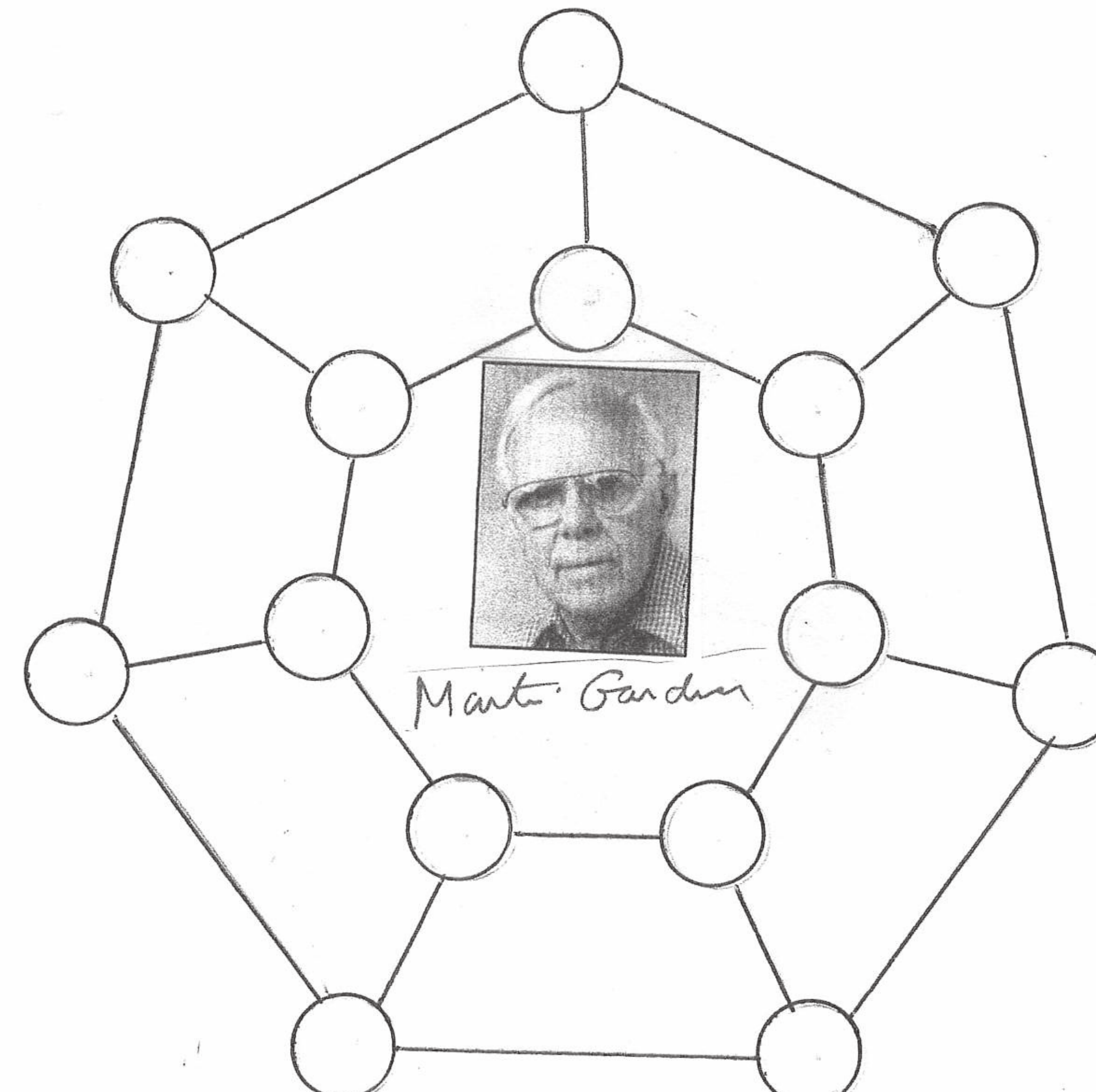
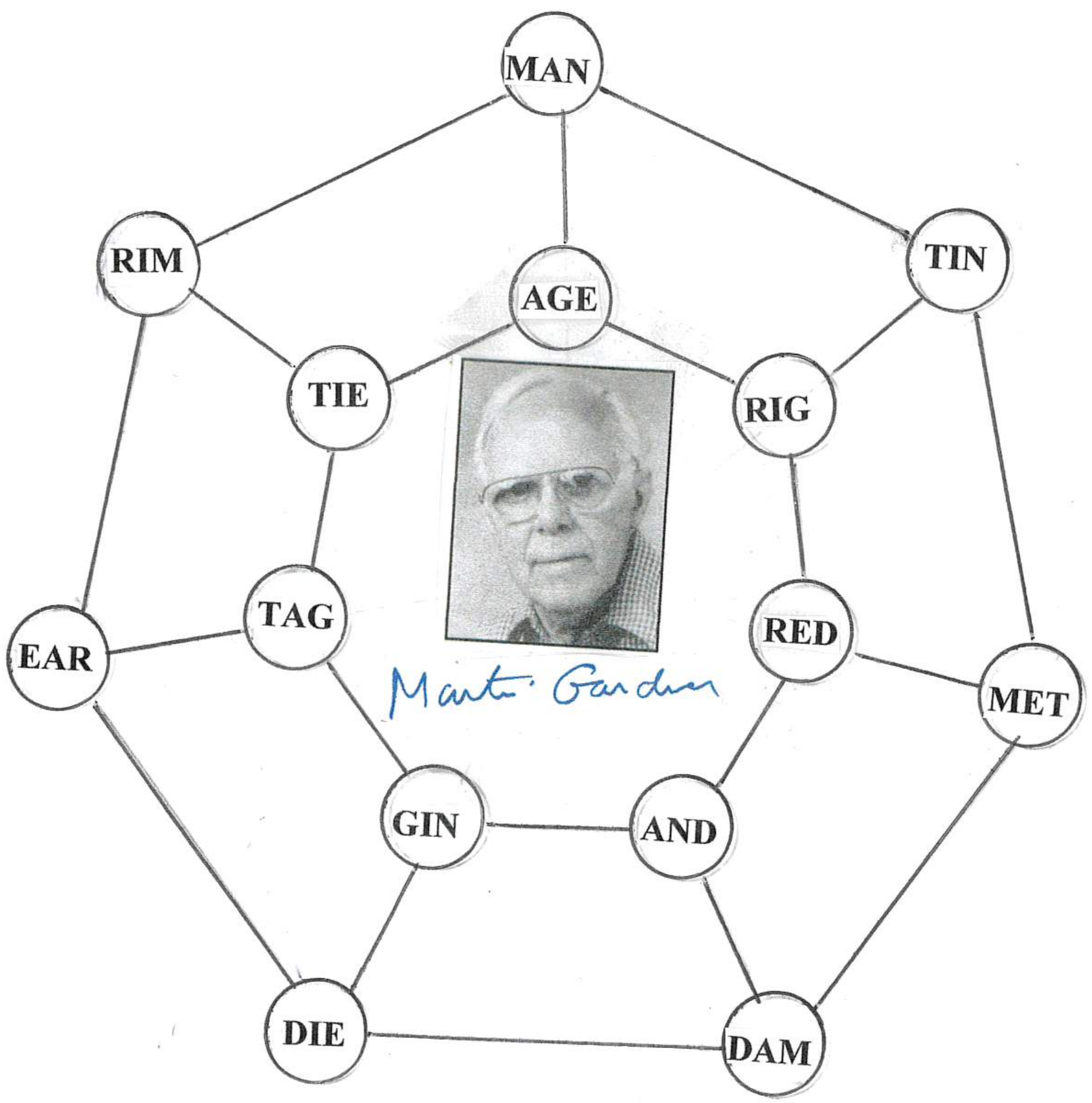
A SCHLEGAL PUZZLE FOR G4G14

by Stephen Bloom and Jeremiah Farrell

This grid is a Schlegal diagram of a solid prism.



As a puzzle place the following 14 words on the nodes so that every two connected nodes have a letter in common. The words are formed from the letters MARTIN GARDNER.
AGE, AND, DAM, DIE, EAR, GIN, MAN, MET, RED, RIG, RIM, TAG, TIE, TIN



Some Toroidal (and Non-toroidal) Rearrangement Puzzles

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In 1998 I came across an etching by American artist Sol LeWitt titled "Straight Lines in Four Directions and All Their Possible Combinations," in an exhibit catalog I found in a used bookstore on a trip to Norman, Oklahoma. (I was there for a conference on tornado forecasting.) The picture, redrawn below, is relatively straightforward: Each of the sixteen squares in a 4x4 array either does or doesn't have a horizontal, vertical, up diagonal or down diagonal line segment drawn in it.

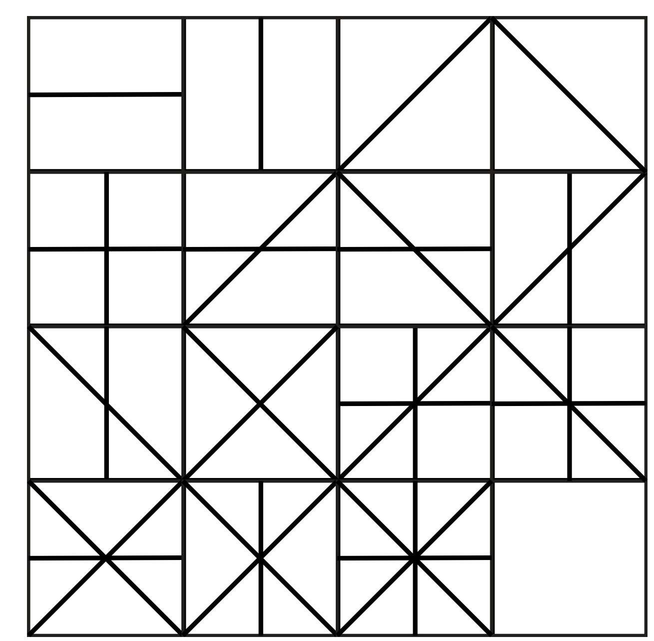


Figure 1: The Sol LeWitt puzzle: 16 squares with all combinations of lines in horizontal, vertical, up-diagonal and down-diagonal directions.

LeWitt arranges the four singlets across the top row, the six doublets across the next row and a half, the four triplets after that, then the square with line segments in all four directions, and finally the “empty” square with nothing at all in it. (Actually, LeWitt’s version does not include the empty square; in the book where I first saw the etching, the title appears there.) LeWitt’s arrangement also draws horizontal lines first, then verticals, then up diagonals, then down diagonals, e.g., the first three doublets are h-v, h-u, and h-d, followed by v-u and v-d, and ending with u-d. I call this kind of systematic approach to laying things out a “Sol LeWitt” arrangement.

My eye (more precisely, my brain…) noticed that some lines continue from one square to another, but rarely all the way across; only a few diagonals make it all the way from one outer edge of the array to another outer edge. This made me wonder: Could the sixteen squares be rearranged, without rotating any of them, into some different 4x4 array so that all lines *do* continue all the way from outer edge to outer edge?

It turns out they can. Not only that, but the solutions have a rather remarkable property: If you move the top row of squares to the bottom, or the left column to the right, you still have a solution. That is, the solutions are all “toroidal” - there's a wrap-around effect, as if the squares were drawn on a donut.

It isn’t surprising – it’s obvious, in fact - that the horizontal and vertical lines in a solution behave toroidally, but it is a surprise that the diagonals do as well; there’s nothing in the question itself that requires it. I eventually wrote this up for an earlier Gathering for Gardner; that paper was published in *Puzzler’s Tribute: A Feast for the Mind* (pp. 387-393).

I’ve tried since to come up with other, similar puzzles whose solutions all have the same toroidal property. So far my efforts have all failed. My first effort appeared in the aforementioned *Puzzler’s Tribute* paper:

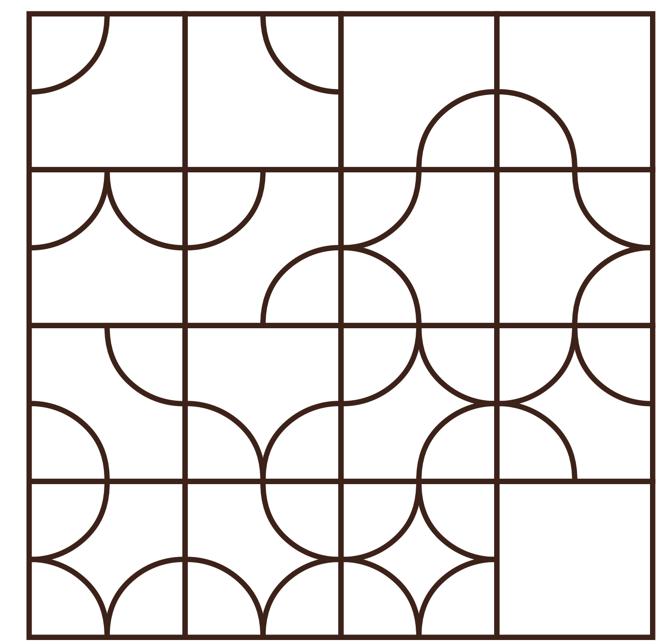


Figure 2: A “Circle” LeWitt puzzle, with quarter circles drawn or not drawn, centered at the four corners of each square.

The design criterion here is that each square either has or doesn’t have a quarter circle centered at each of its four corners, and the problem is to rearrange the given 4x4 array of squares, sans rotations, so that each arc continues from square to square. There are 32 quarter circles in all, so a toroidal solution will have 8 complete circles (some of which appear as pairs of semicircles on opposite sides of the array). This puzzle does have toroidal solutions, but it also has solutions that are non-toroidal, so in that sense it’s a failure.

I later tried a variant I called the “Sine" LeWitt problem: Along each edge of each square, either do or don’t draw a half period of a sine wave, and then try to rearrange the squares, again without rotations, so that you get sets of sine curves running from left to right and top to bottom. (Note, there’s not much difference, visually, between a half sine wave and a quarter circle, especially when they’re hand drawn. This opens the possibility for an alternative puzzle in which the goal is to arrange the squares so that each quarter circle is part of a complete circle.)

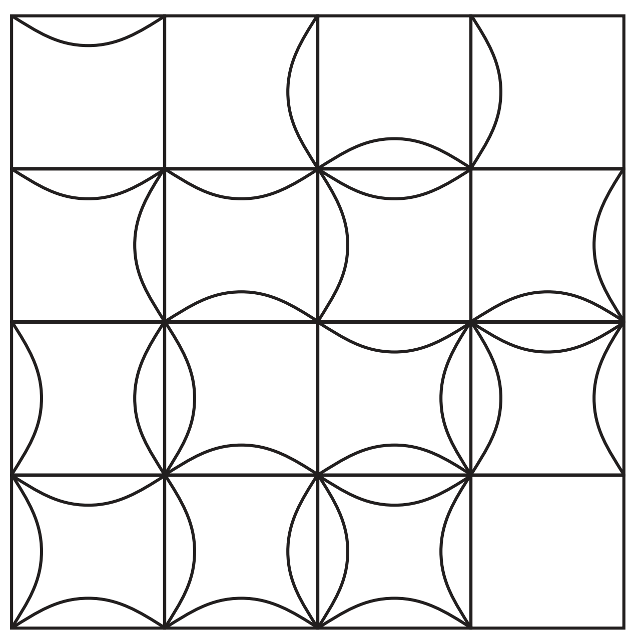


Figure 3: The “Sine" LeWitt puzzle.

But like the Circle LeWitt puzzle, Sine LeWitt has both toroidal and non-toroidal solutions, so it’s another failure.

Before I go on, a word about rotations: I self-imposed the non-rotation rule mainly to keep the sixteen squares all different. If, for example, you rotate the Sol LeWitt square with a single

up diagonal by a quarter turn, it becomes a duplicate of the down-diagonal square. (Some squares, of course, don’t change if you rotate them by a quarter turn, and all squares are invariant under half turns.) I usually label the squares in my designs in a way that subtly discourages rotations. But if you want to rotate pieces, go right ahead. Just know, it’s a somewhat different problem then. In particular, if you allow rotations, the original Sol LeWitt problem has additional solutions that are *not* toroidal.

Recently, in 2019, I decided to turn the whole problem on its head, and designed a set of sixteen different squares for which no matter how you arrange them, you get continuity from square to square, with toroidality understood to occur at the outer edges:

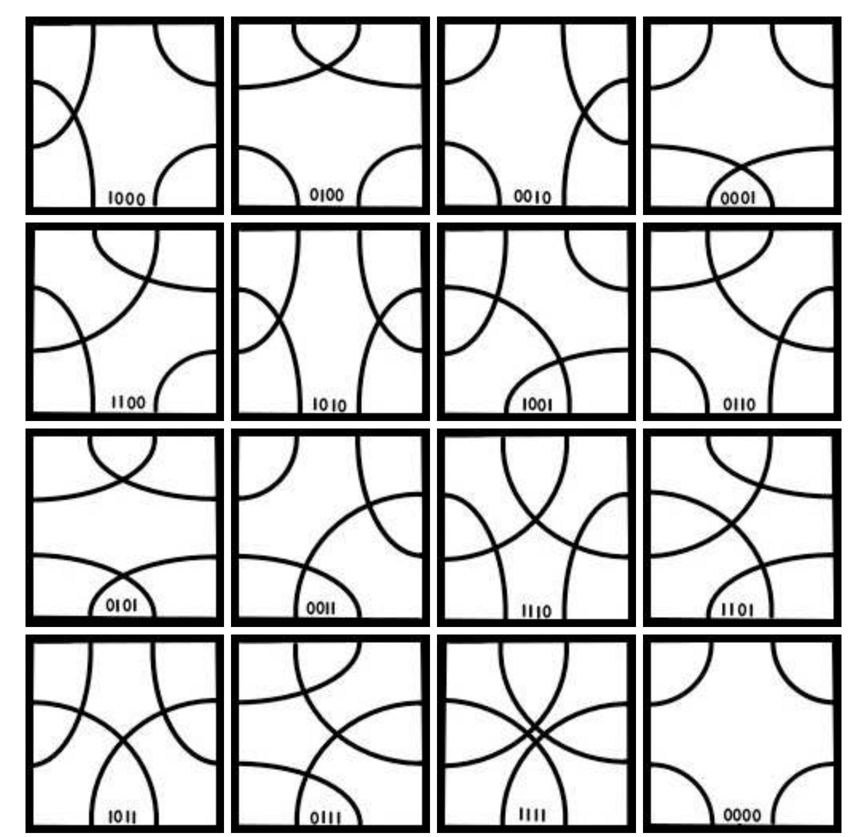


Figure 4: A toroidal looping puzzle, with labels to discourage rotation of squares, in “Sol LeWitt” order, with single-crossing squares first, then two-crossing squares, etc.. (Figure courtesy of Donna Dietz – see http://www.donnadietz.com/cipra/CipraPuzzle.html for a playable version of the puzzle.)

Each square has four arcs in it, with each arc connecting two “thridpoints” (an invented term for midpoints that divide an interval into thirds) of adjacent edges; the key rule that limits the number of different patterns to 16 is that the two arcs emanating from the thridpoints on each edge must connect to thridpoints on *opposite* edges. The four-bit label in each square specifies whether the two arcs emanating from the thridpoints of the left, top, right, and bottom sides of the square, in that order, do or do not cross, with “1” if they do and “0” if they don’t. One of the labels’ roles is to discourage rotation, but you are, as before, welcome to refuse to be discouraged, and rotate to your heart’s content.

A note about “thridpoints”: It’s a purely aesthetic choice to divide each side of the squares into thirds; any two points on each pair of sides will do; the important thing is that arcs continue from one square to the next. Indeed, one way to discourage rotations would be to choose “thridpoints” asymmetrically, so that continuations would be disrupted if any of the squares were rotated. It’s also an aesthetic choice to use quarter circles and quarter ellipses for the arcs; the essential property is continuity, not smoothness. An interesting question to ponder is whether aesthetic choices enhance the process of mathematical discovery or restrict it – or both!

It occurred to me later to incorporate a consistent rule for passing one arc over another wherever there’s a crossing, which makes the set of squares, if you “fatten” the arcs so they look like stretches of string, look like this:

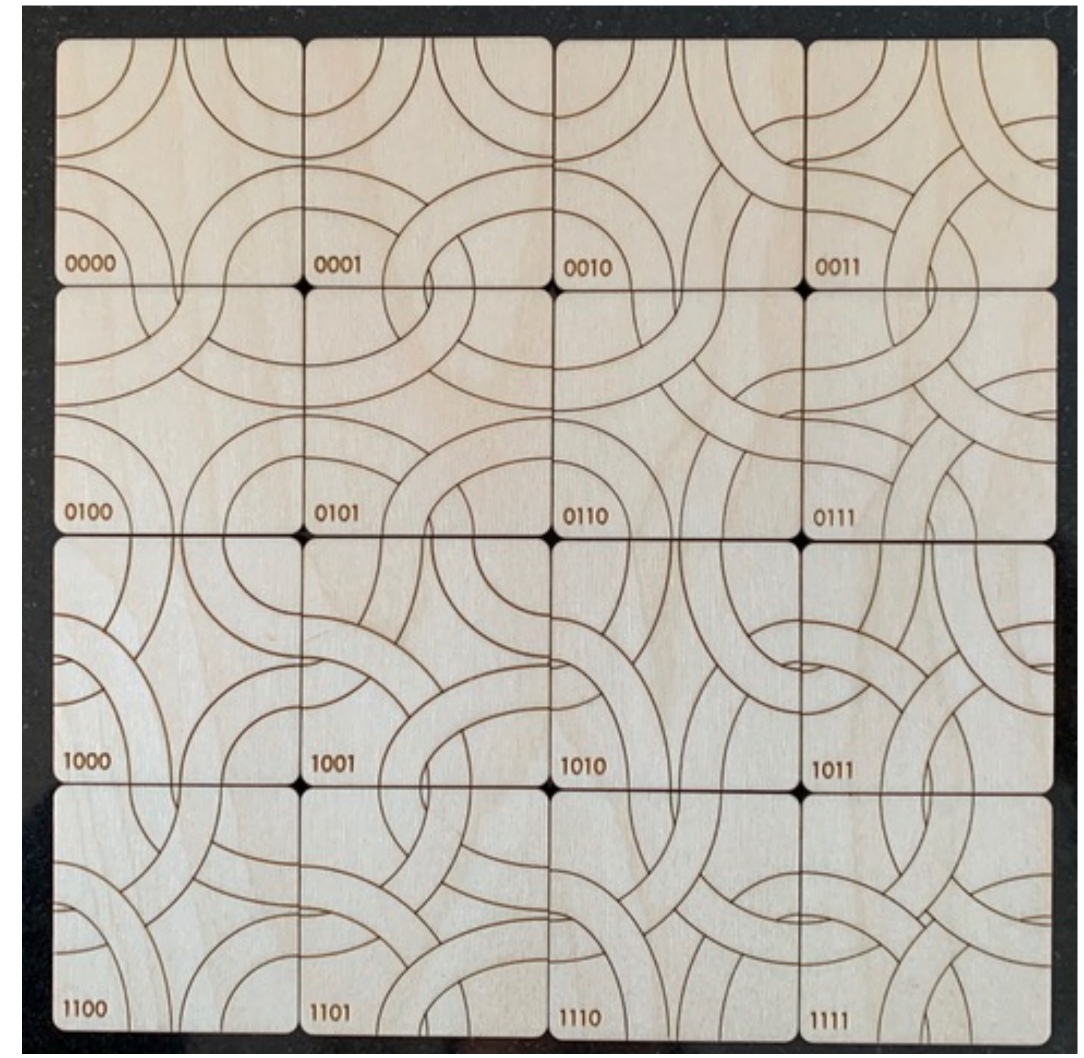


Figure 5: Toroidal looping puzzle “fattened” into over- and under-passes, arranged in "binary" fashion, with labels ordered from 0000=0 to 1111=15. (Photo courtesy of Pete Benson at CherryArborDesign.com - the puzzle can be purchased there.)

One pleasant surprise here is that, no matter how you rearrange the squares – and even if you allow rotations – the sequence of over- and under-crossings always alternates. Experts in knot theory undoubtedly see this as obvious; the rest of us can be content to scratch our heads or work out an ad-hoc proof.

I debuted this puzzle at the 2019 MOVES conference at the National Museum of Mathematics in New York, without specifying the particular puzzle I had in mind for the pieces. You might notice I haven't done so here either (yet). I did so in part to see what ideas others would come up with for what could be done with the pattern. I invite readers to pause at this point and think for themselves of something interesting to do with the pieces. (At MOVES I did not even hint that the pattern should be cut into separate squares; some people came up with the idea of cutting, but along the *arcs*, like a jigsaw puzzle.)

One person at MOVES (I'm sorry, I don't remember who it was) observed that the pieces looked like "Tsuro" tiles, named after a popular board game of relatively recent vintage. Tsuro tiles also connect the "thridpoints" on the four sides of a sqare, and some of them are identical with the tiles in my puzzle, but others are not. I'm not sure what rule (if any) governs the set of Tsuro tiles; as mentioned above, I chose a rule that produces exactly 16 different patterns.

So here's the challenge I had in mind when I invented the puzzle: Can you rearrange the tiles so that there is exactly one loop that runs through all the arcs of all the squares?

Since each tile has four arcs, there are 64 arcs in all. It's convenient to talk about the "length" of a loop as the number of separate arcs it consists of. If you patiently count them, you will find that the "binary" arrangement in Figure 5 has four

loops, each of length 16. The "Sol LeWitt" arrangement in Figure 4 has a pair of loops of length 4 that are fairly easy to spot; the rest of its arcs belong to two loops, each of length 28.

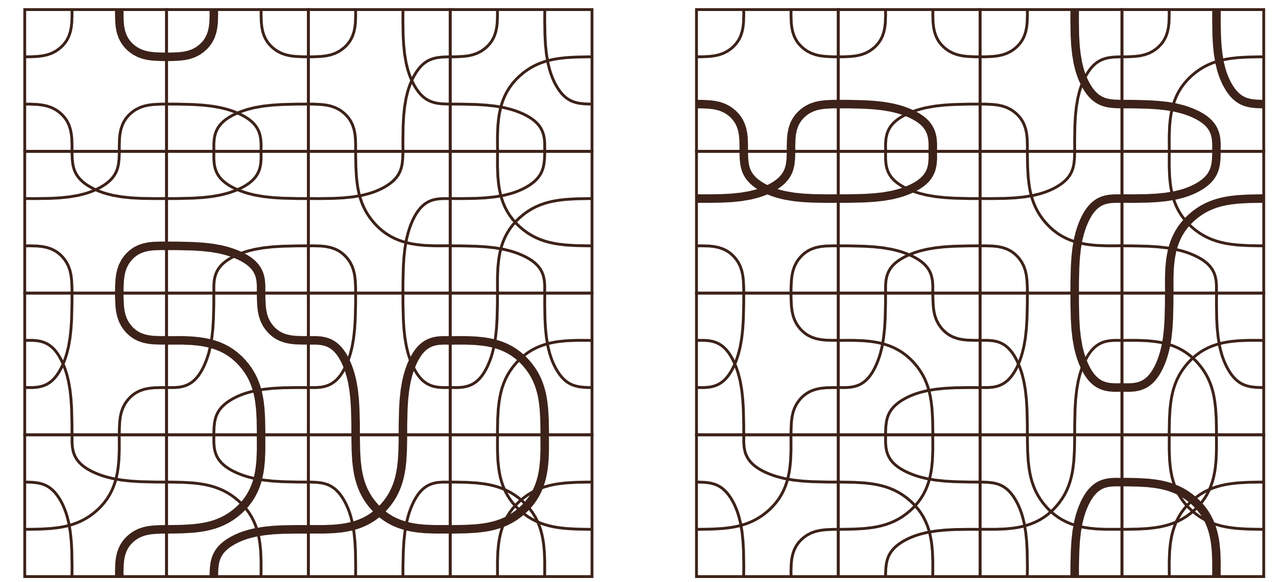


Figure 6. Two loops, each of "length" 16, in the "binary" arrangement from Figure 5. (Note, the arcs here are poorly drawn quarter circles and ellipses, as evident from a careful look at the bottom rightmost tile.)

Notice that all those loop lengths are multiples of 4. It's not hard to see that loop lengths must be even; a two-color checkerboard proof does the trick: Each loop passes back and forth between black and white squares. To show the number of arcs is a multiple of 4, use a four-coloring of 2x2 patches, say rows of alternating Red/Blue alternating with rows of alternating Green/Yellow (hence columns of alternating Red/Green alternating with columns of alternating Blue/Yellow). If, in following a loop, you pass from Red to Blue, you'll next pass from Blue to Yellow no matter which way you turn (up or down), then from Yellow to Green, then from Green back to Red, after which you'll wind up repeating the color sequence again and again.

Peter Winkler took an interest in the puzzle at MOVES; by the end of the afternoon he had a proof that a single, 64-arc loop is impossible. A key observation was that all attempts finding a one-loop arrangement invariably left an *even* number of loops. What Peter finally proved was that, no matter what set of tiles you use (i.e., let each square be any tile, even if you repeat some tile patterns multiple times and not use others at all), the parity of the number of loops is equal to the parity of the number of 1's in the tiles' labels. Donna Dietz, who was also at the MOVES conference, wrote up Peter's proof, along with other observations the three of us made, in a paper posted on the ArXiv: https://arxiv.org/abs/1908.05718 . She also posted a playable version of the puzzle on her website, as noted in Figure 4. (Clicking on any two squares there interchanges them, so you can move pieces wherever you want.)

Peter's proof works for any even-by-even array of my tile patterns; it doesn't work if one of the dimensions is odd.

Jim Propp, another MOVES attendee, had an interesting suggestion at the meeting: Instead of toroidal connections, whenever an arc came to the outer edge of the 4x4 square array, connect it to the arc in the nearest neighboring square, with three-quarter-circle connections at the four corners. For the initial "binary" array, you get this:

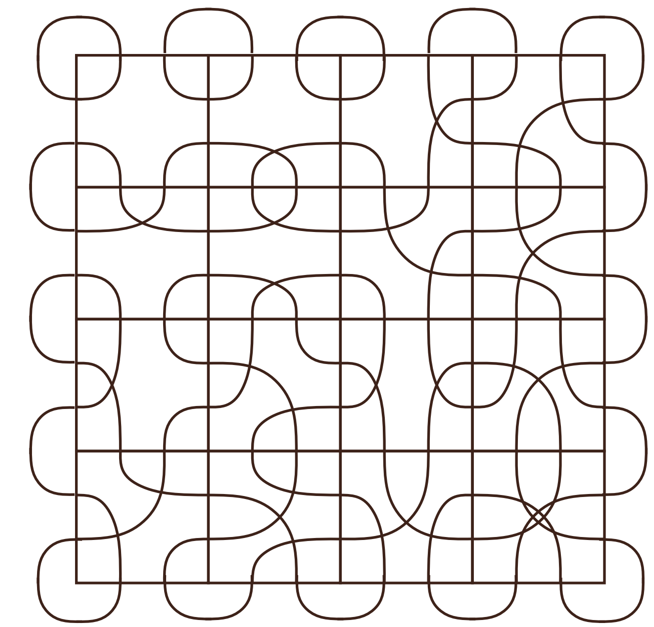


Figure 7: Jim Propp's non-toroidal suggestion for the looping puzzle problem.

When Jim showed me this, I suggested connecting adjacent thridpoints *within* each edge around the perimeter instead:

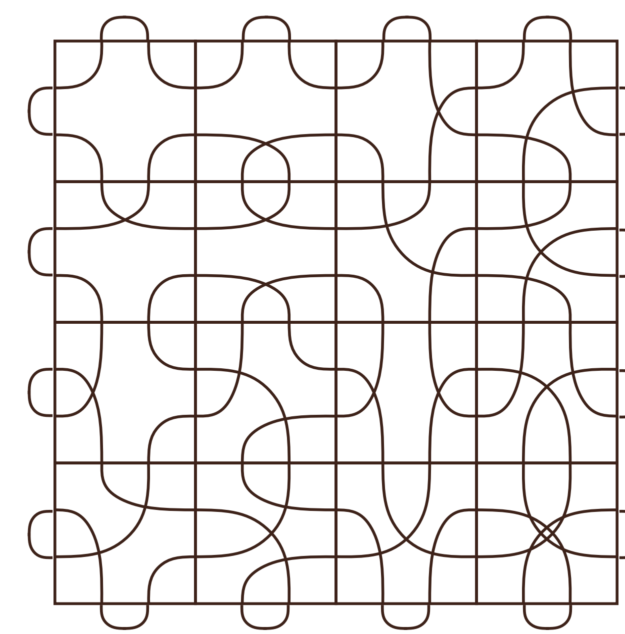


Figure 8: My alternative to Jim Propp's non-toroidal suggestion for the looping puzzle problem.

To our considerable surprise, this *does* consist of one single loop! It’s still unclear, to me at least, if there’s anything behind this beyond mere happenstance.

Later, in some email correspondence, when he saw the over- and under-passing version of the looping puzzle, Jim complained that the 16 tiles were no longer a complete set of possible patterns: There should really be a separate tile for each assignment of which arc goes under the other when two arcs cross. This would lead to a 9x9 puzzle with a total of 81 different tiles. That’s a bit big for my taste, but I urge anyone undaunted by the size to see if there’s anything of interest it.

In response to Jim’s complaint, I designed a 4x4 “Toroidal Trellis” problem:

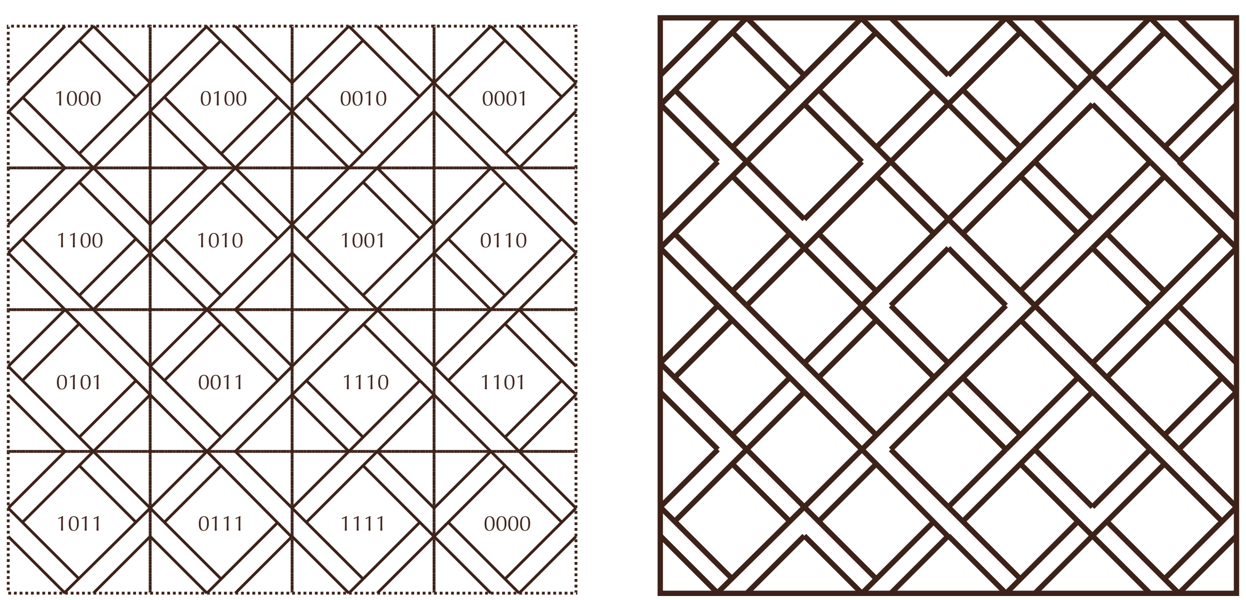
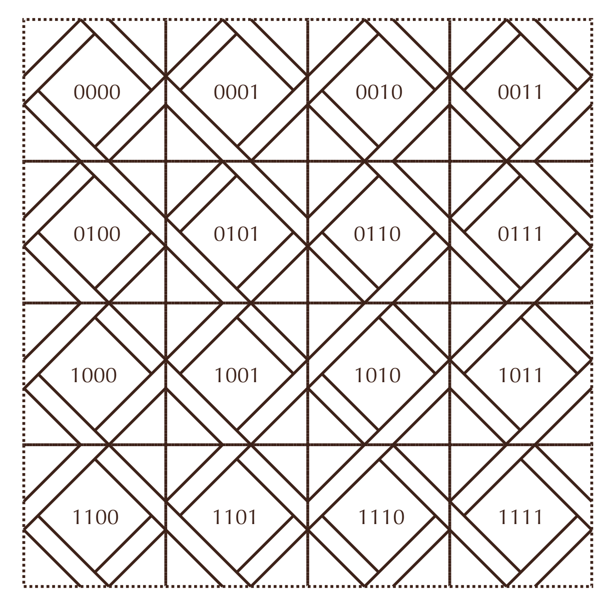


Figure 9: A 4x4 “Trellis” puzzle, with squares in a “Sol LeWitt” arrangement, shown with labels and guidelines (left) and as pure trellis (right).

The idea is to think of each tile as containing four thin slats of wood, running diagonally by quarter turns, with one slat lying over the other where they meet at the tile’s edge. The binary numbers indicate whether the slat “entering” an edge (in a clockwise direction) lies over or under the slat “exiting" the edge, starting at the tile’s topmost edge. One can now picture the pattern as a trellis, by joining the “upper slats” that meet from the two sides of each edge and likewise for the “lower” slats. Since there are 64 slats altogether, one can again ask if there’s an arrangement of the tiles so that the trellis, again with toroidal connections at the outer edges of the 4x4 array, consists of a single loop. I again don’t know the answer.

Alternative to toroidal identifications, one can imagine the pattern in a 4x4 arrangement as a ribbon that reflects with a crease when it hits an outer edge of the array. Amazingly, the “Sol LeWitt” arrangement in Figure 9 above *is* a single loop! So are the 4x4 “binary” arrangement and a “magic square” arrangement:



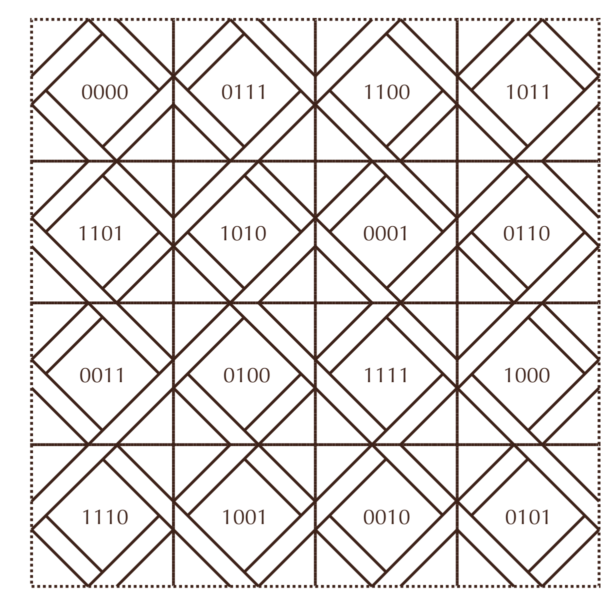


Figure 10: The Trellis puzzle in its “binary" arrangement (left) and a “magic square arrangement (right). Viewed as (non-toroidal) ribbons creased at the outer edges of the array, each is an example of a single loop.

Certainly not *every* arrangement of the nontoroidal “ribbon” trellis consists of a single loop, since it’s easy to arrange the tiles so as to produce short loops of length 4. But I suspect that a large number of arrangements do give a single loop. My only evidence of this, however, is the fact that the first time I tried a “random” arrangement, it turned out to be single-looped. Since there are 16! = 20,922,789,888,000 different non-toroidal ways to lay out the 16 tiles, I’d be surprised indeed if I just got lucky. It might be worth someone’s time counting the exact number of single-loop arrangements. (It might be worthwhile doing the same for Jim Propp's and/or my non-toroidal versions of the looping puzzle problem.)

More recently, in 2021, I got to wondering if I could reduce the size of the looping puzzle from 4x4 to 3x3. That is, could I come up with a set of *nine* patterns that exhaust all the possibilities for some design criterion? (It also occurred to me to see if I could reduce things yet further to a 2x2 version of a puzzle. The ultimate, of course, would be to come up with a challenging 1x1 puzzle!) The problem is that 9 doesn’t easily relate to 4. But what finally occurred to me is that among the 24 permutations of four objects, exactly 9 are *derangements*, i.e., permutations that have no fixed elements. So this suggested two pairs of possibilities:

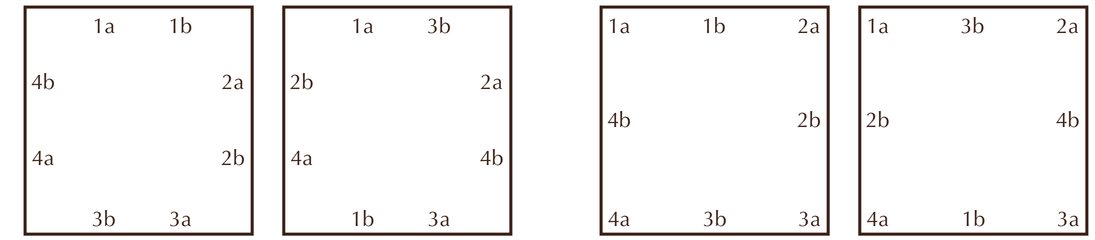


Figure 11: Two labellings of the thridpoints of a square (left) and two labellings of the corners and midpoints (right) that lend themselves to a “deranged” looping puzzle. The idea is to connect each “a” point to a “b” point with a *different* number.

The “thridpoint” and corner-midpoint labellings in Figure 11 produce these two sets of 9 different tiles:

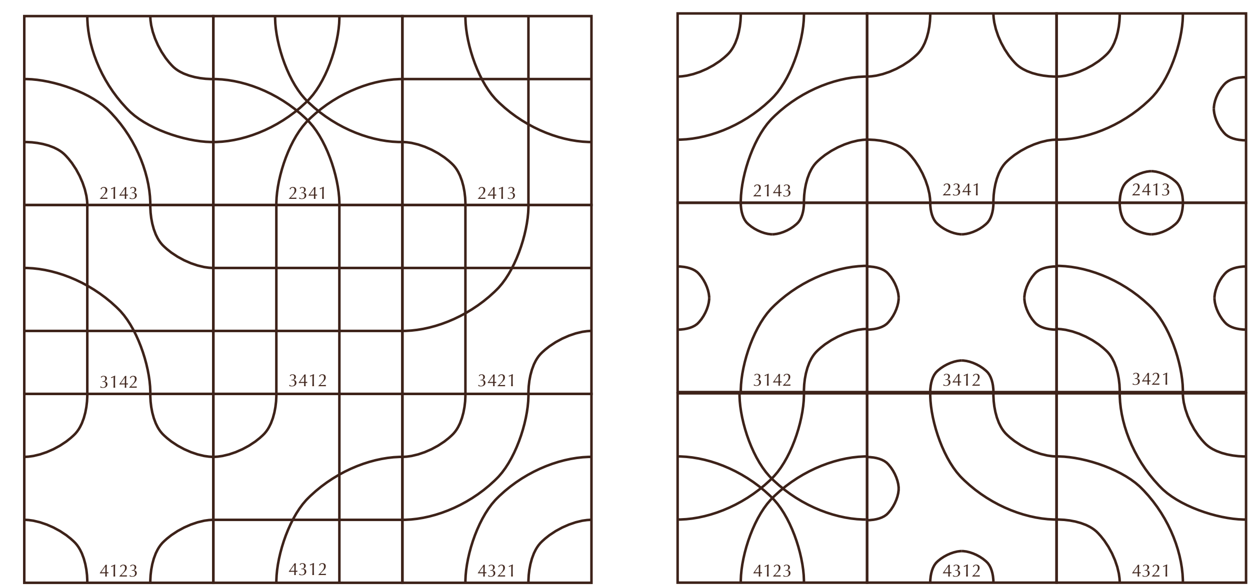


Figure 12: Two toroidal “derangement" puzzles based on connecting thridpoints as labeled in Figure 11 (left). Can either of these be rearranged so as to have a single toroidal loop?

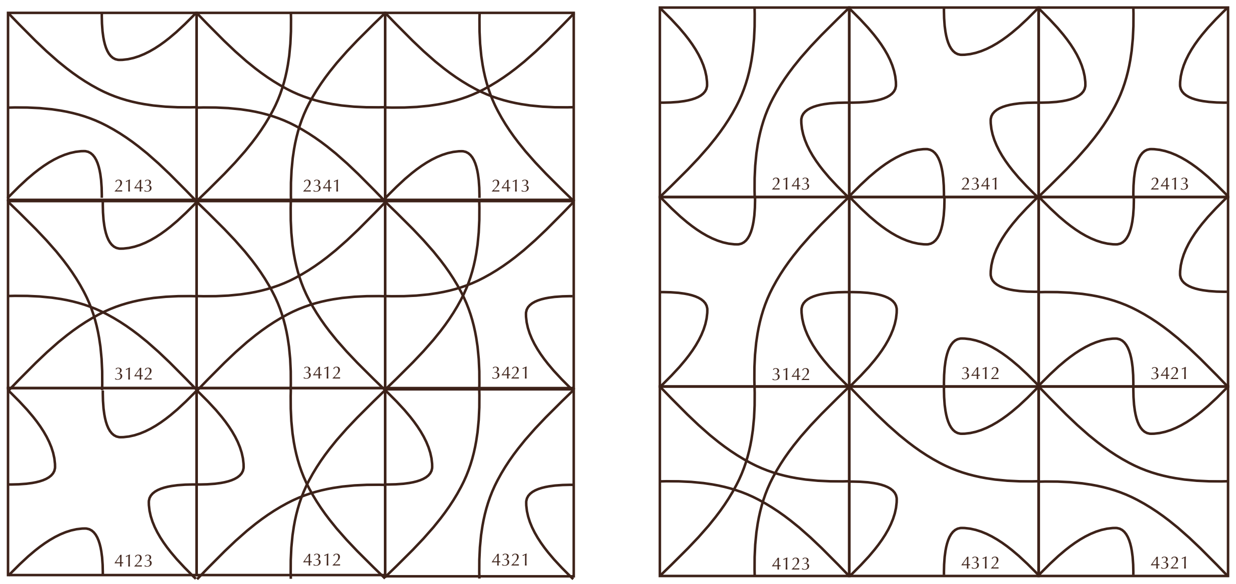


Figure 13: Two toroidal “derangement" puzzles based on connecting corners to midpoints as labeled in Figure 11 (right). Can either of these be rearranged into a single toroidal loop? (Note, the convention at corners is to continue from one square into the diagonally adjacent square.)

These final four puzzles are small enough that a brute-force (computer) search could easily resolve them: There are, after all, only 8! = 40,320 toroidally different arrangements. (The 4x4 puzzles have 15! = 1,307,674,368,000 toroidally different arrangements, which is big enough to call for some clever pruning.) I myself have not spent any time, either brute-force or cleverly, looking systematically for arrangements that give a single loop, but I have noticed that for two of the puzzles, randomly rearranging the squares often produces a single-loop solution, whereas for the other two I’ve yet to find a single-loop arrangement. I leave it to the reader to guess (and then check) which two are which – and, ideally, to figure out why.

Puzzle Fonts About Puzzles

Erik D. Demaine* Martin L. Demaine*

Abstract

We present five recent puzzle fonts — where reading glyphs in the font require solving a puzzle — that illustrate five different puzzles/puzzle games, each with a corresponding mathematical result. Each font is an open-source interactive web application that lets the user write messages in the font, and then solve the resulting puzzles (or send them to a friend to solve), revealing the message.

1 Introduction

We have been developing a growing series of mathematical and puzzle fonts — which recently reached 30 different typefaces[1] — that you can interact with in web apps.[2] Every one of these fonts is ***mathematical*** in the sense that it illustrates a mathematical theorem or open problem. Most of our typefaces also offer one or more ***puzzle*** fonts, where reading the text requires solving a mathematical puzzle. These fonts were recently featured in *The New York Times* [Rob21].

In this paper, we describe five recent typefaces that share the theme of both having puzzle fonts and being *about* puzzles. Figure 1 gives a visual overview. The first typeface is about a puzzle video game, Tetris, while the other four typefaces are about pencil-and-paper puzzles: Sudoku, Yin-Yang, Path Puzzles, and Tatamibari.

The presentation corresponding to this paper is available on YouTube.[3] All fonts presented here, including the slides and code that generates all shown figures, are free and open source, with code available on GitHub.[4]

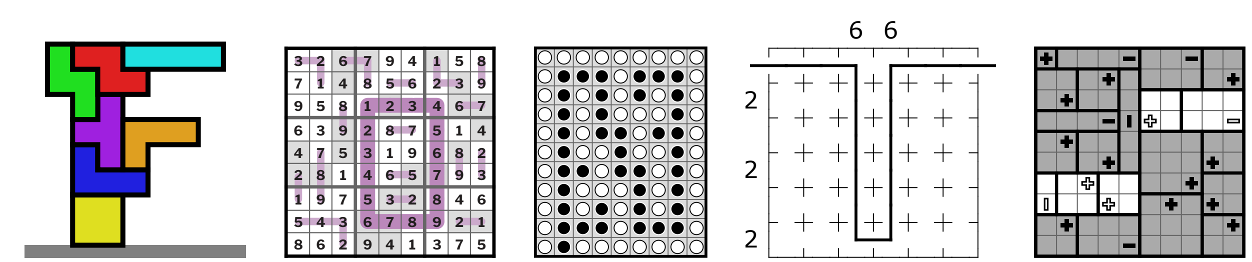


Figure 1: FONTS written in the five puzzle typefaces described here: Tetris (Section 2), Sudoku (Section 3), Yin-Yang (Section 4), Path Puzzles (Section 5), and Tatamibari (Section 6).

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[1]A ***typeface*** is a collection of multiple related ***fonts***.

[2]https://erikdemaine.org/fonts/

[3]https://youtu.be/K6M3ELHr5Ls

[4]https://github.com/edemaine/talk-puzzle-fonts-about-puzzles/

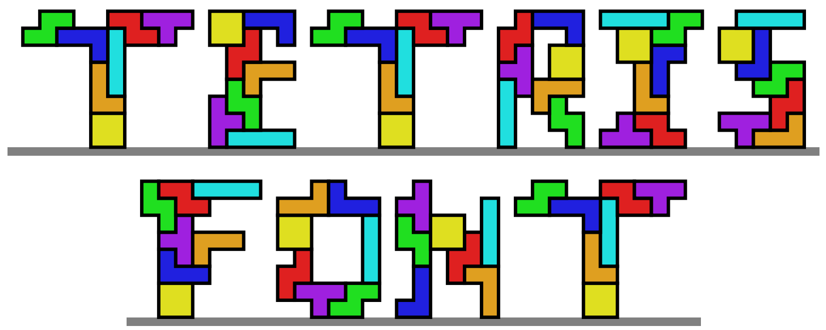

Figure 2: TETRIS FONT written in the Tetris solved font.

2 Tetris

The ***Tetris typeface***[5] represents each letter by a stacking of a complete set of oriented tetrominoes, that is, one of each of the possible pieces in the Tetris video game. Figure 3 shows the stackings for the entire alphabet. Each stacking is designed to be executable in Tetris physics, with pieces stacked in order and stopping when they hit a previously placed piece. Figure 4 shows how the pieces can be ordered so that their falling in sequence produces the letters in Figure 3. The web app offers an animated font that simulates Tetris gameplay.

Displaying all the pieces splayed out in fall order, as in Figure 4, is one way to make puzzles with a Tetris puzzle font. For example, can you read the secret messages in Figures 5 and 6?

Another way to make puzzles with the Tetris font is to hide the individual pieces, and ask the viewer to figure out how the Tetris pieces exactly tile the letter-shaped regions. These packing/tiling problems can be quite challenging; when developing the font, we made extensive use of the BurrTools software[6] which can solve such puzzles by brute force.

We presented this font in a paper that proved a new mathematical result about Tetris: the perfect-information game is NP-complete even with just 8 columns or 4 rows [ACD+20].

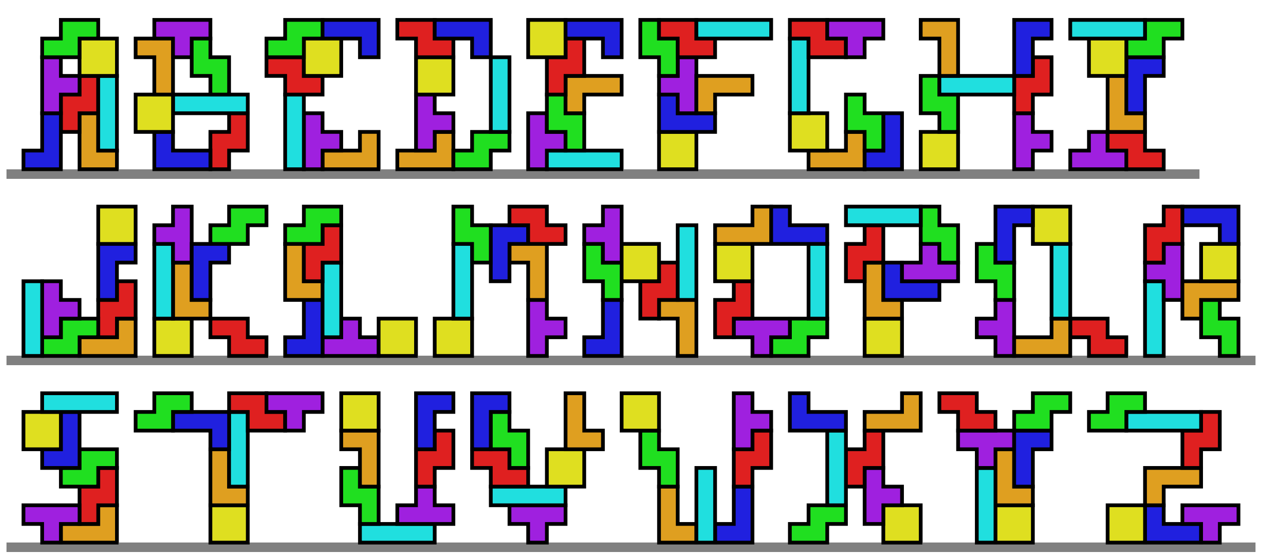

Figure 3: The entire alphabet in the Tetris solved font.

[5] https://erikdemaine.org/fonts/tetris/
[6] http://burrtools.sourceforge.net/

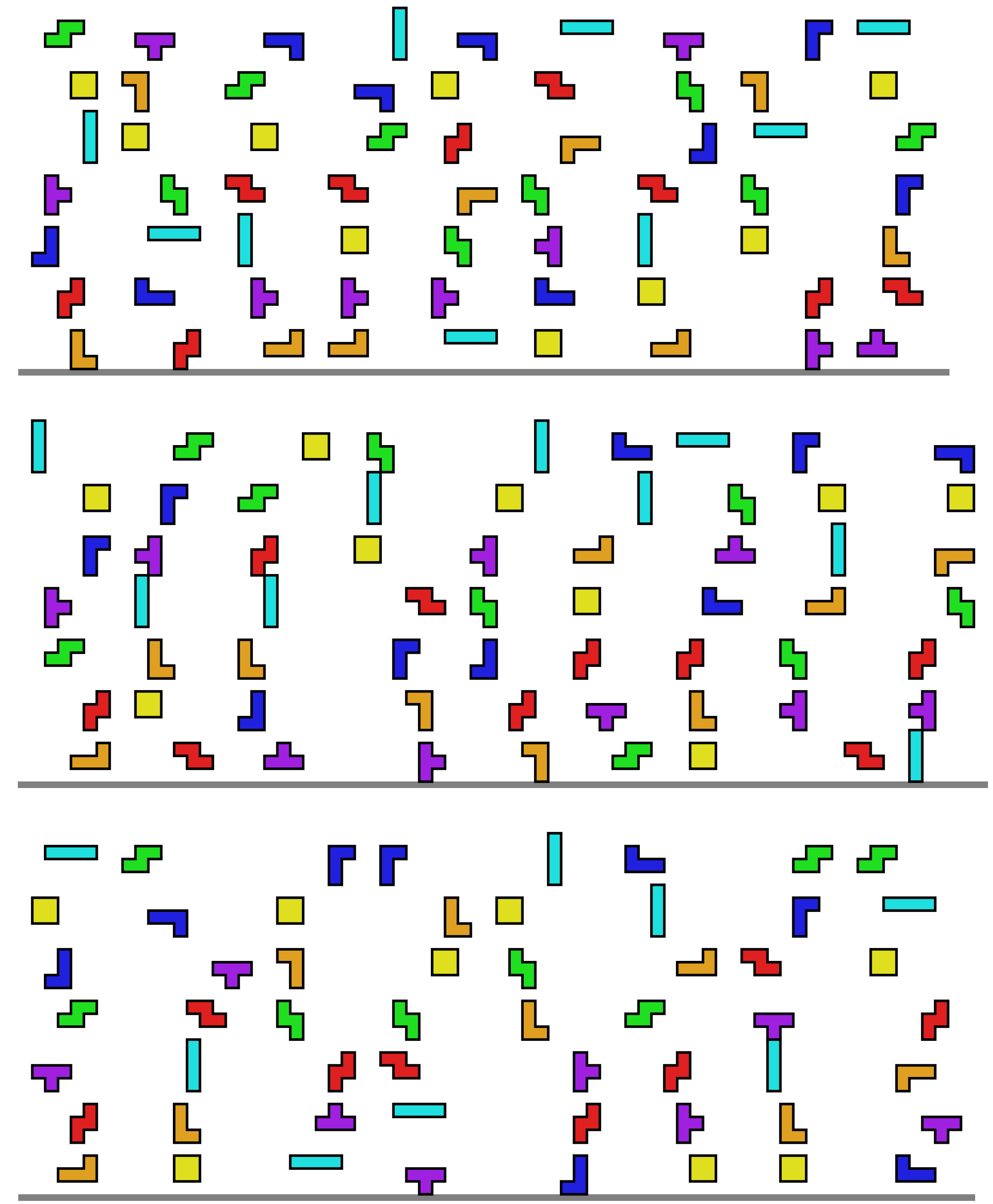

Figure 4: The entire alphabet in the Tetris falling-puzzle font.

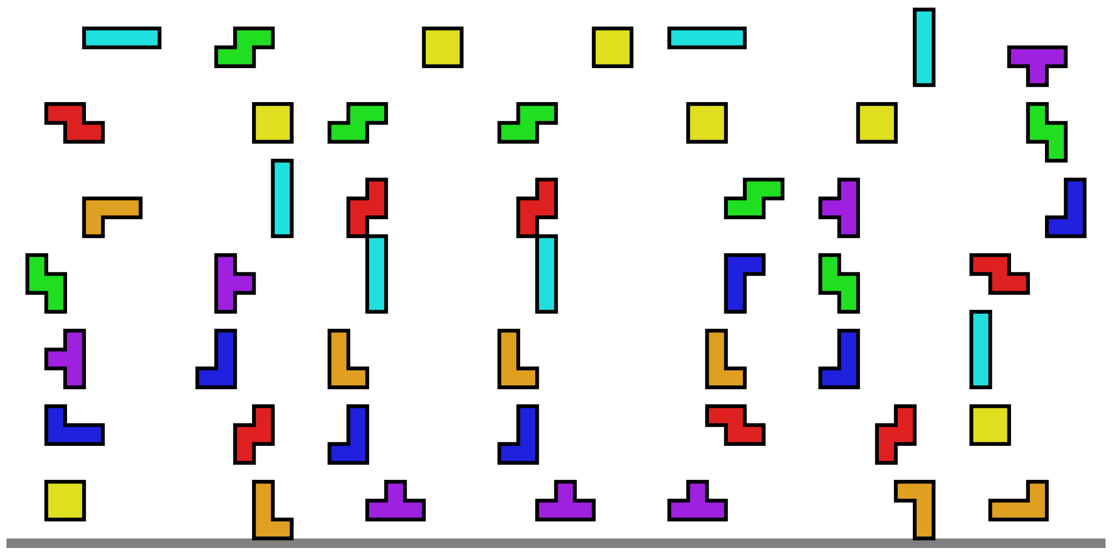

Figure 5: What message do you get if each piece falls straight down until it hits one of the other pieces (or the floor)? The solution is in Figure 25.

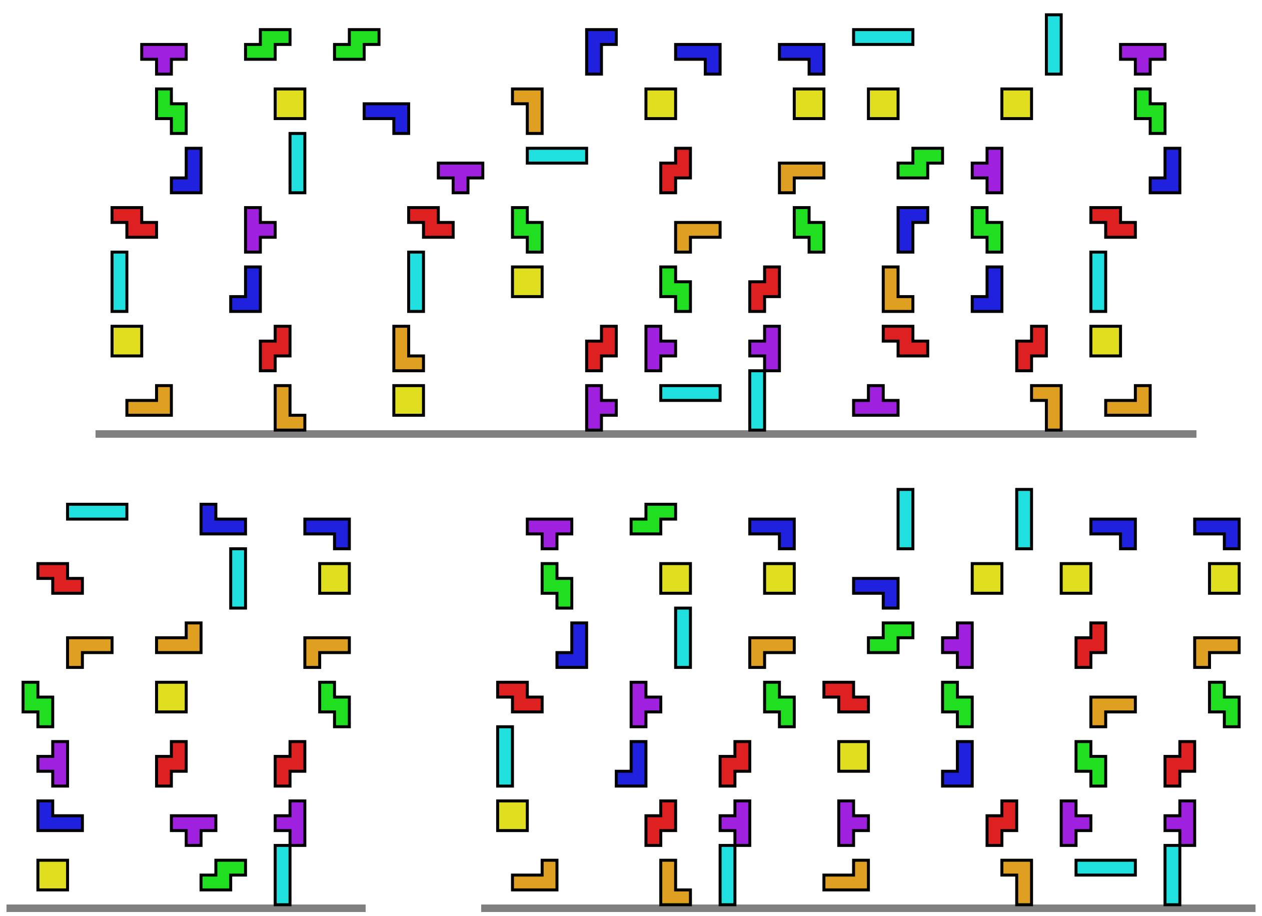

Figure 6: What message do you get if each piece falls straight down until it hits one of the other pieces (or the floor)? The solution is in Figure 26.

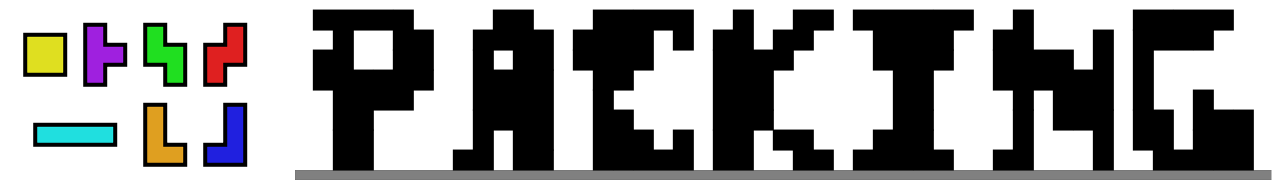

Figure 7: Can you tile each letter with exactly the pieces on the left? Solutions are in Figure 27.

Figure 8: Can you tile each letter with exactly the pieces on the left? Solutions are in Figure 28.

3 Sudoku

The ***Sudoku typeface***[7] draws a letter of the alphabet in a Sudoku puzzle, by connecting consecutive numbers in the solution (connecting all 1s to 2s, all 2s to 3s, etc.), and drawing the longest path among these connections. Figure 9 shows examples of puzzles and their solutions, and Figure 10 shows the full alphabet. We designed the intended paths by hand, and used our own brute-force computer search to find $81 = 9 \cdot 9$ compatible solutions. Then we reduced each solution to a corresponding minimal puzzle that can be uniquely solved by a human (without lookahead) by randomly removing locally derivable clues.

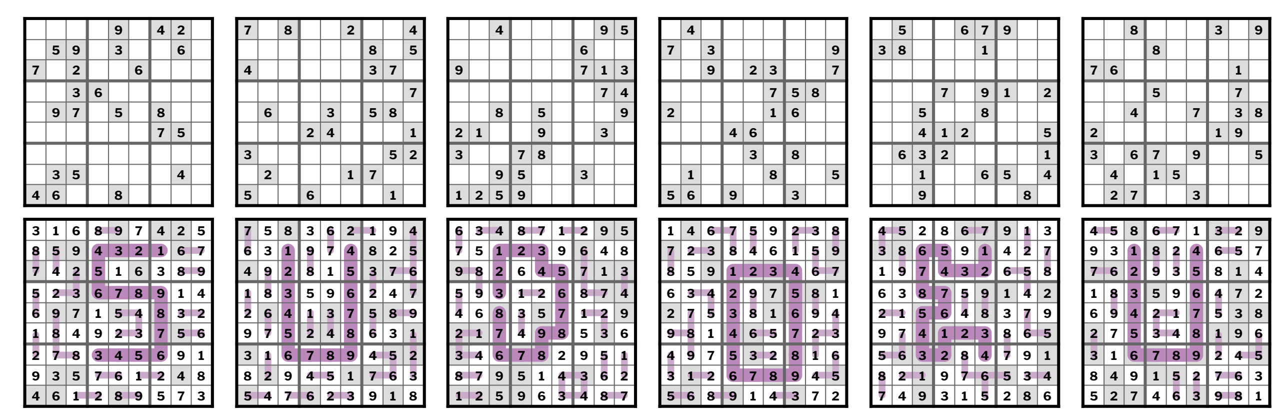

Figure 9: SUDOKU written in one of the 81 Sudoku puzzle fonts (top) and the corresponding solved font (bottom). The bottom figure highlights connections between consecutive numbers, with thick lines denoting the longest path of such connections.

[7] https://erikdemaine.org/fonts/sudoku/

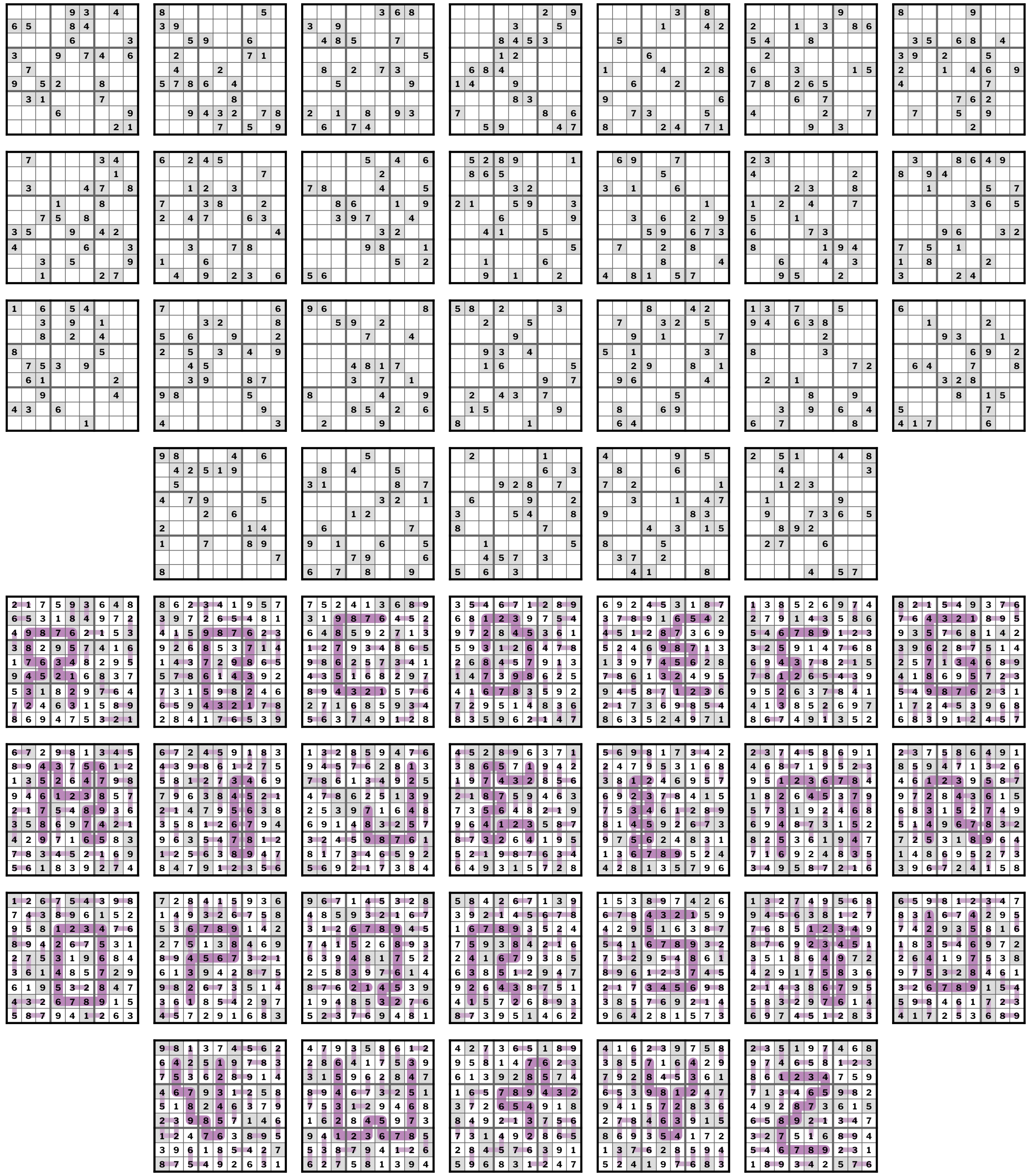

Figure 10: The entire alphabet written in one of the 81 Sudoku puzzle fonts (top) and the corresponding solved font (bottom).

The 81 Sudoku puzzle fonts let us hide messages in Sudoku puzzles. Figures 11 and 12 give two puzzles for you two try. Alternatively, try solving some puzzles on the interactive web app.

Sudoku also has a corresponding complexity result: it is NP-complete and, even stronger, ASP-complete [YS03]. In fact, finding the longest path among a square grid of connections is also NP-complete [IPS82].

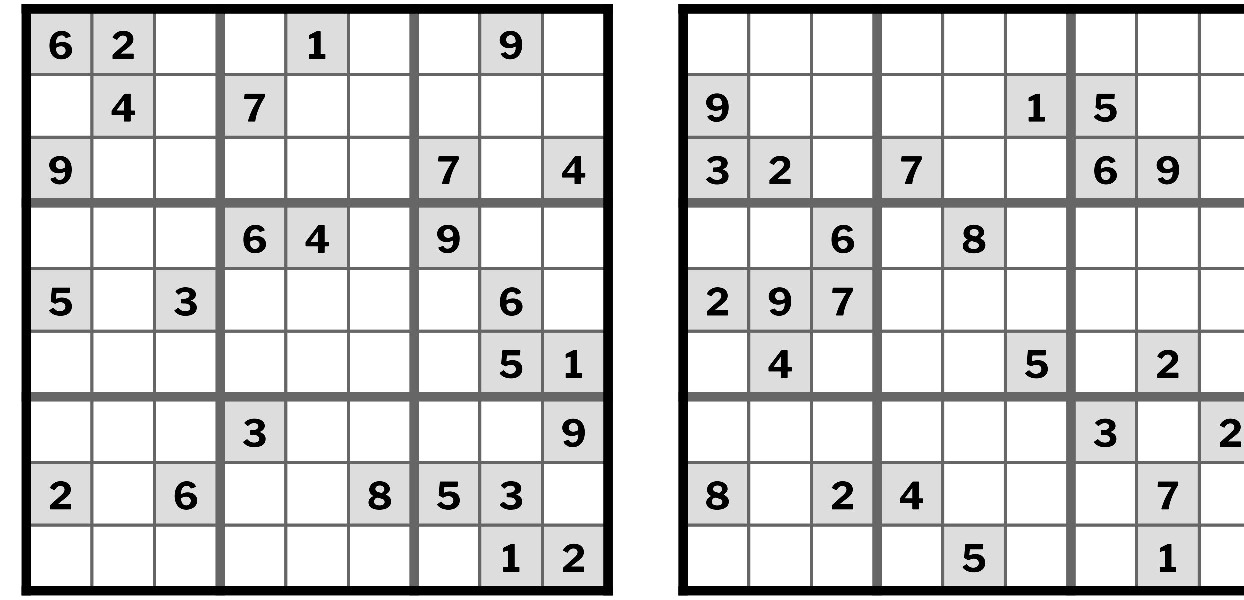

Figure 11: Can you solve the Sudoku puzzles, connect consecutive numbers, and find the longest paths, to reveal the hidden message? Or solve interactively on the web. The solution is in Figure 29.

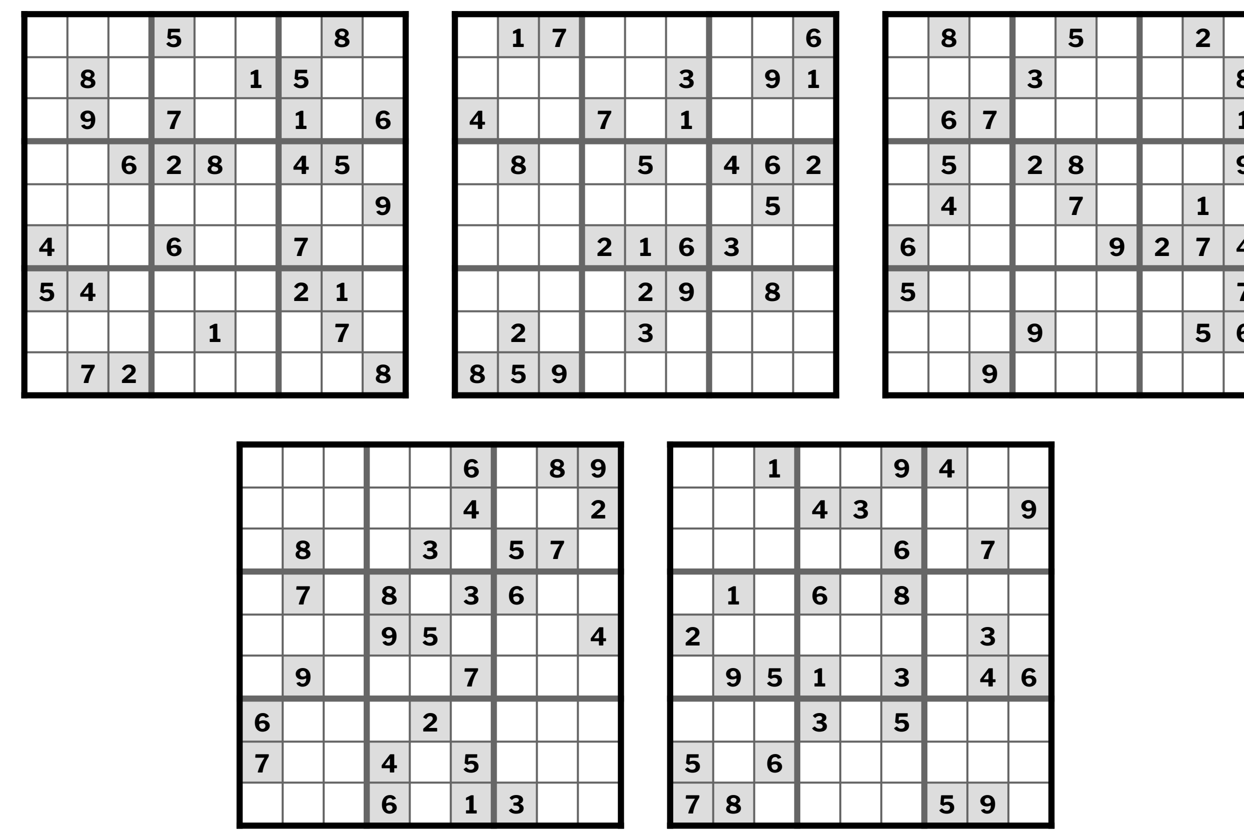

Figure 12: Can you solve the Sudoku puzzles, connect consecutive numbers, and find the longest paths, to reveal the hidden message? Or solve interactively on the web. The solution is in Figure 30.

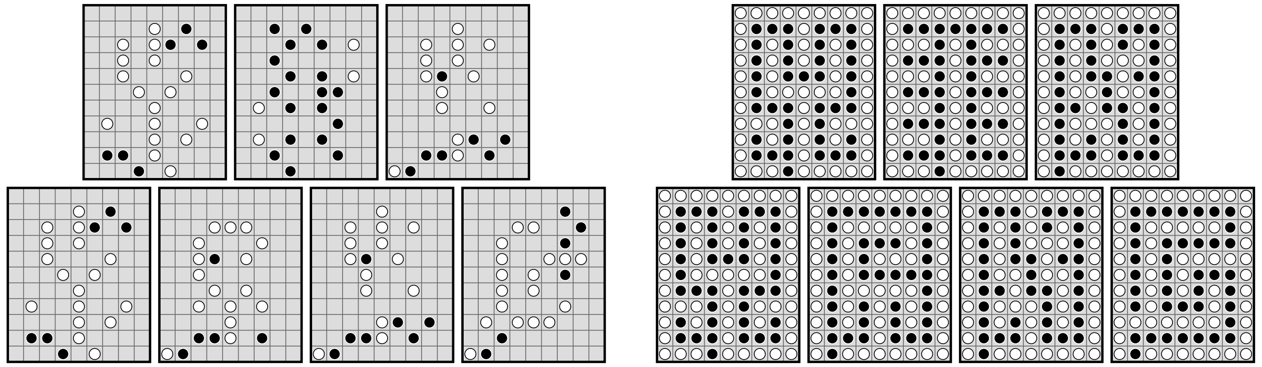

Figure 13: YIN YANG written in the Yin-Yang puzzle font (left) and solved font (right).

4 Yin-Yang

The ***Yin-Yang typeface***[8] represents each letter of the alphabet by another type of pencil-and-paper puzzle called ***Yin-Yang***. The puzzle is on a square grid, with some of the squares prefilled with a black or white circle. The goal is to fill in the remaining squares with black and white circles so that (1) the black circles are connected by horizontal and vertical connections, (2) the white circles are similarly connected, and (3) there are no 2×2 squares with circles of the same color. Figure 13 shows examples of puzzles and their solutions, and Figure 15 shows the full alphabet. In each case, the black circles outline the letter.

We designed the solutions by hand, then used our own brute-force computer search to repeatedly remove clues that preserved unique solvability, resulting in a minimal puzzle with the intended solution. After hundreds of such trials, we hand-picked what seemed to be the most challenging puzzle for each letter. Nonetheless, some letters (such as V) are relatively difficult, while others (such as E, J, and M) are relatively easy.

As usual, the puzzle font lets us hide messages in the puzzle. Figures 14 and 16 give two such messages for you to try. The font is also designed to make it possible to combine multiple letters into a single puzzle (while still satisfying the constraints), though the resulting minimal puzzles seem to be substantially easier to solve. Figure 17 shows an example.

Along with this font, we proved that Yin-Yang puzzles are NP-complete [DLRU21].

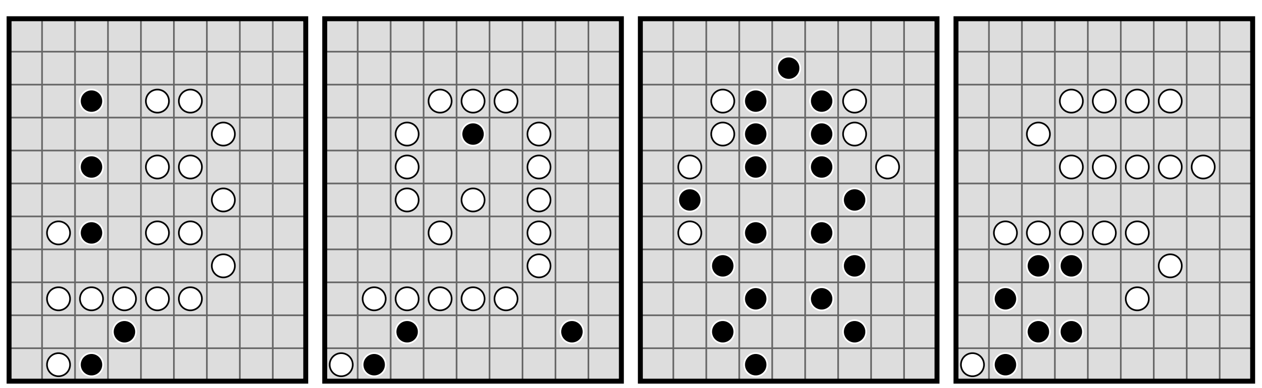

Figure 14: Can you connect together the black and white dots without a monochromatic 2×2 square? Or solve interactively on the web. The solution is in Figure 31.

[8] https://erikdemaine.org/fonts/yinyang/

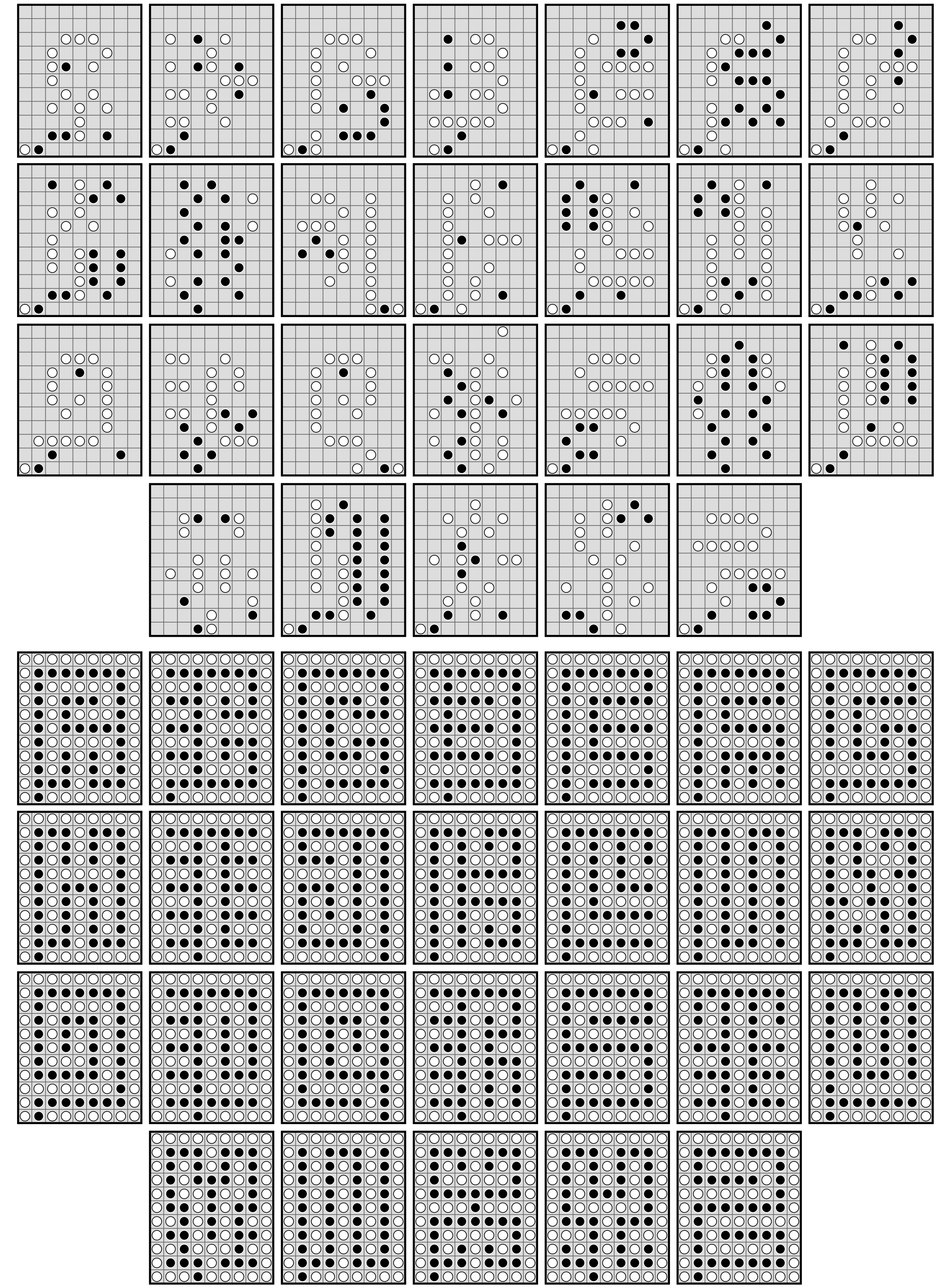

Figure 15: The entire alphabet written in the Yin-Yang puzzle font (top) and solved font (bottom).

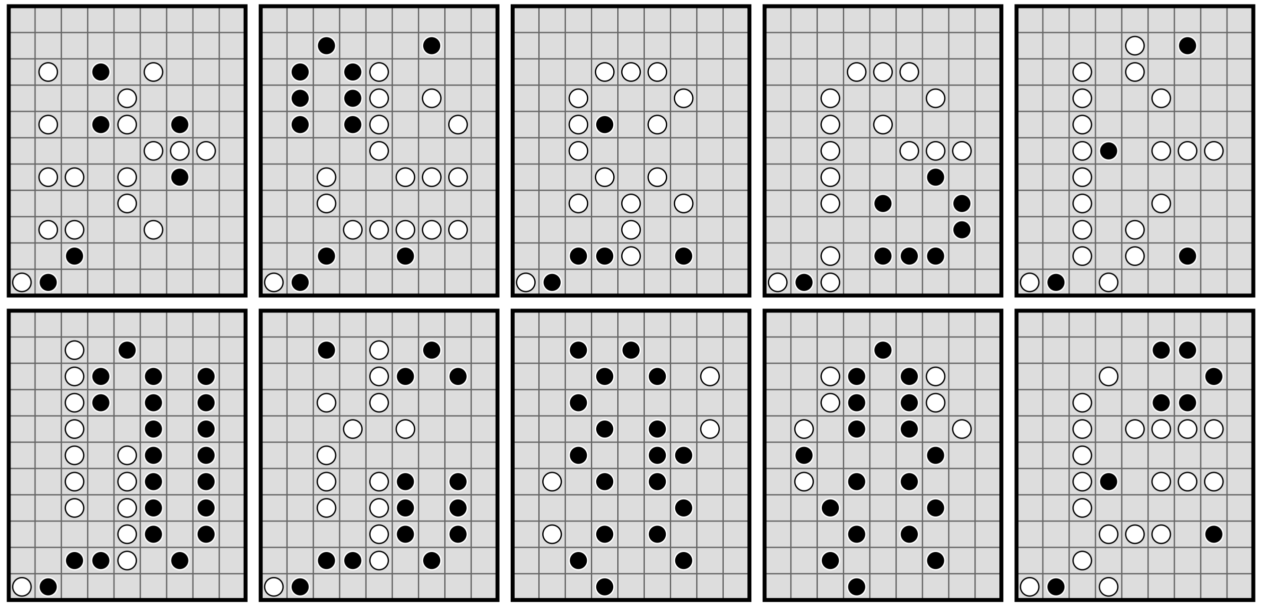

Figure 16: Can you connect together the black and white dots without a monochromatic 2×2 square? Or solve interactively on the web. The solution is in Figure 32.

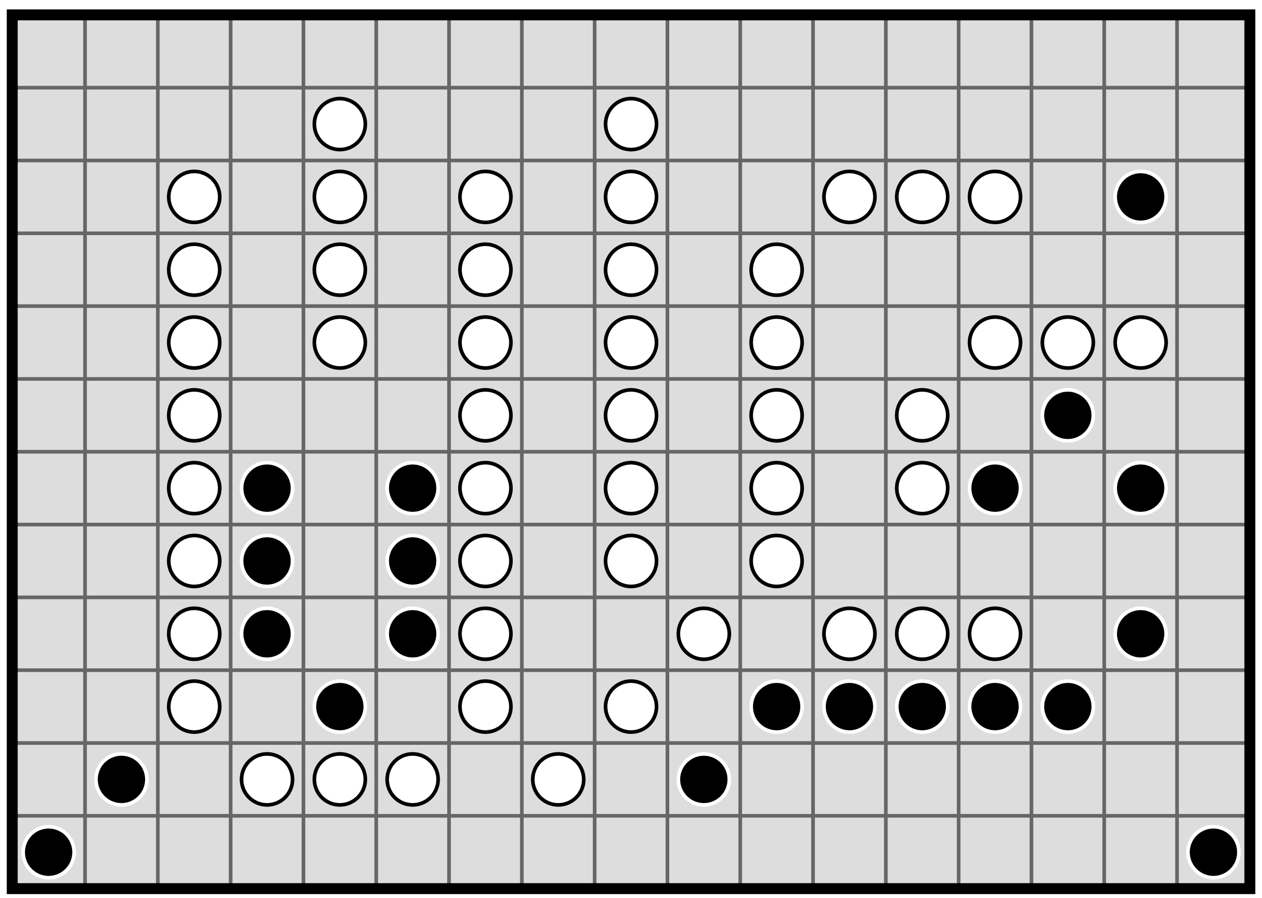

Figure 17: Can you connect together the black and white dots without a monochromatic 2×2 square? Or solve interactively on the web: https://erikdemaine.org/fonts/yinyang/g4g.html. The solution is in Figure 33.

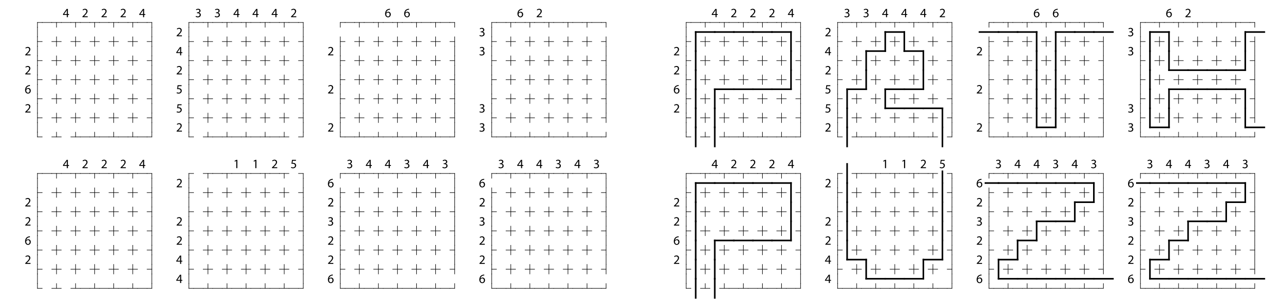

Figure 18: PATH PUZZ written in the path-puzzles puzzle font (left) and solved font (right).

5 Path Puzzles

The ***path-puzzles typeface***[9] represents each letter of the alphabet by another type of pencil-and-paper puzzle called ***path puzzles***. The puzzle is on a square grid with two gaps on the boundary, where some rows and some columns are marked with an integer. The goal is to draw a single path between the two gaps such that the number of filled squares in each row and column matches the marked integer (if given). Figure 18 shows examples of puzzles and their solutions, and Figure 20 shows the full alphabet. In each case, the path draws the letter.

These puzzles were designed by hand to have unique solutions, and verified to have unique solutions by our own brute-force computer search, in a larger team. We presented this font in a paper that proved NP-completeness of path puzzles [BDD+20].

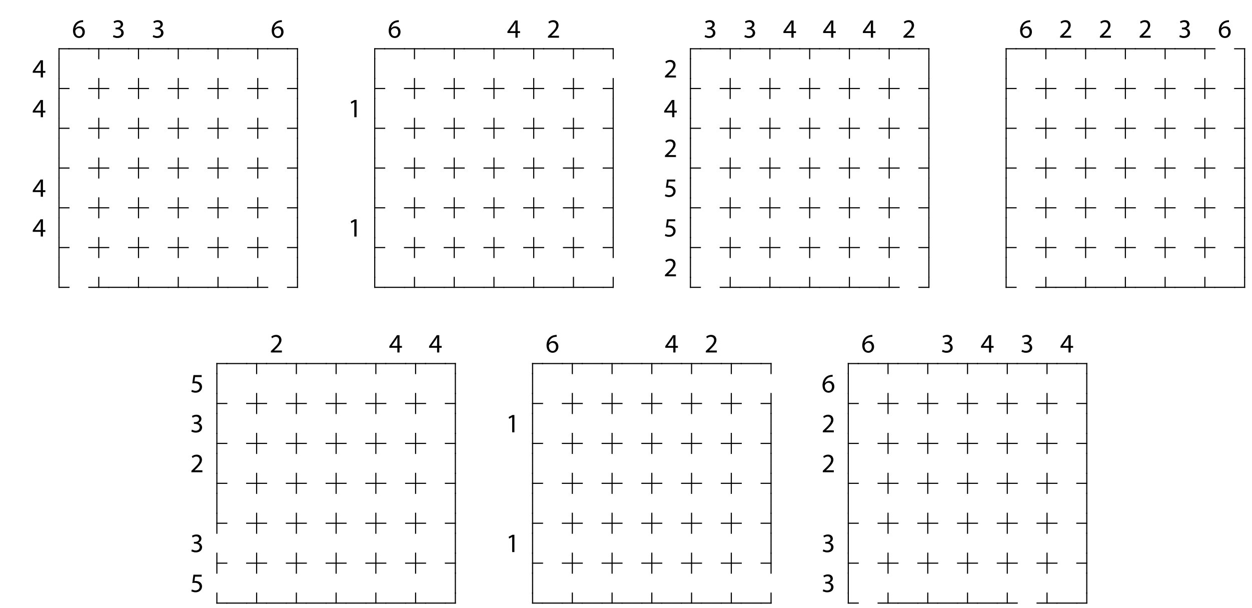

Figure 19: Can you draw a path between the two boundary gaps that has the specified numbers of filled squares in indicated rows and columns? Or solve interactively on the web. The solution is in Figure 34.

[9] https://erikdemaine.org/fonts/pathpuzzles/

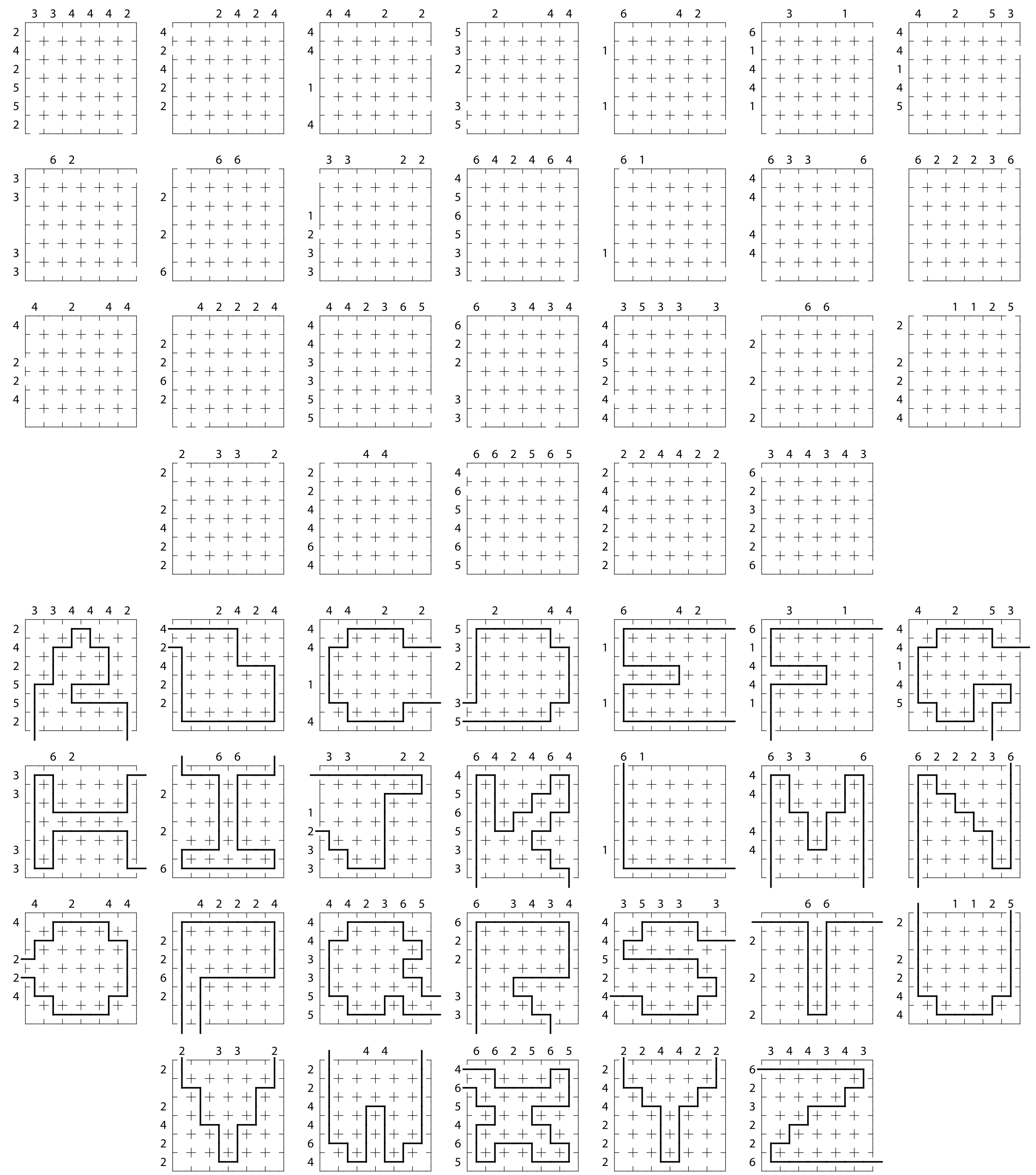

Figure 20: The entire alphabet written in the path-puzzles puzzle font (top) and solved font (bottom).

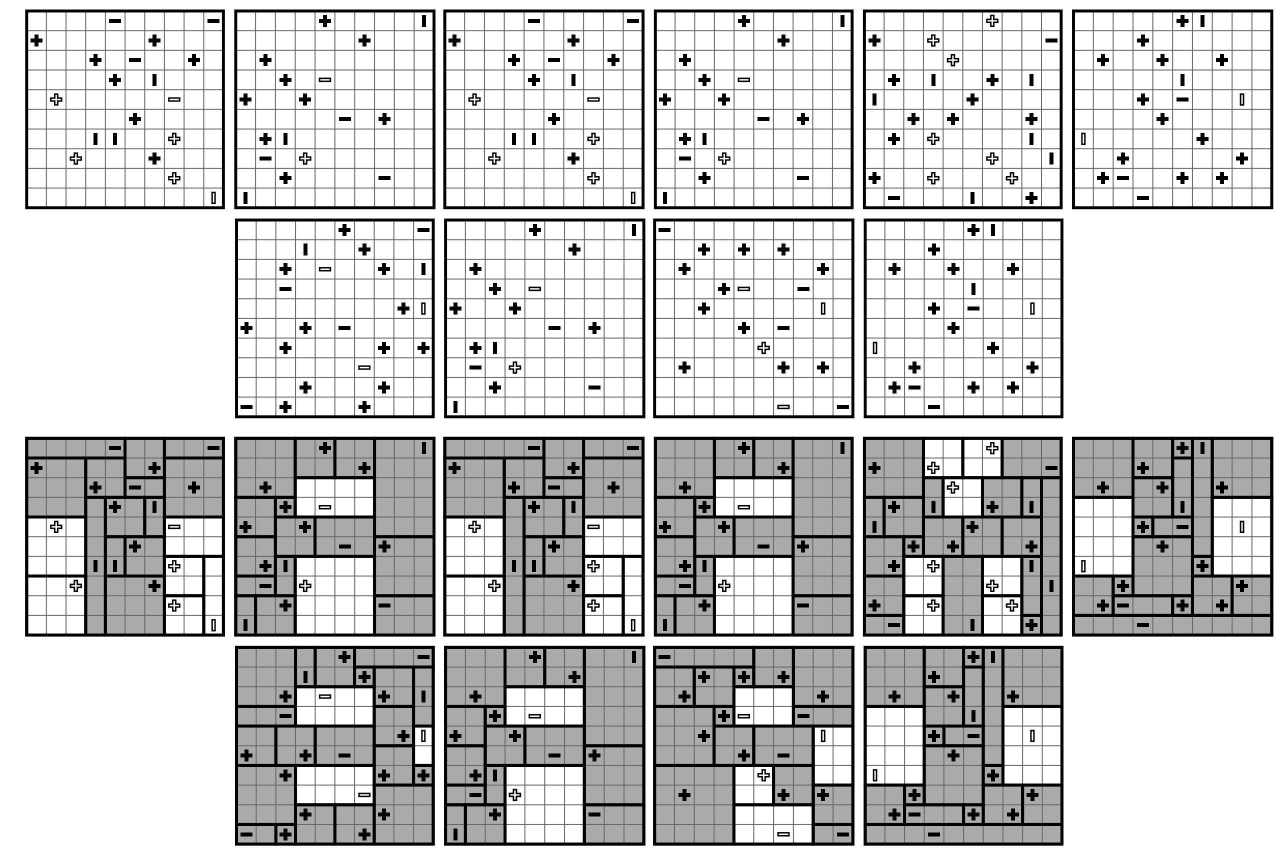

Figure 21: TATAMIBARI written in the Tatamibari puzzle font (top) and solved font (bottom).

6 Tatamibari

The ***Tatamibari typeface***[10] represents each letter of the alphabet by a final type of pencil-and-paper puzzle called ***Tatamibari puzzles***, published by the famous Japanese puzzle publisher Nikoli. The puzzle is on a square grid, with some cells marked with a clue in the shape of a plus sign, horizontal bar, or vertical bar. The goal is to decompose the grid into exactly one rectangle per clue such that (1) each plus clue is in a square; (2) each horizontal clue is in a nonsquare rectangle that is wider (more horizontal) than it is tall; and (3) each vertical clue is in a nonsquare rectangle that is taller (more vertical) than it is wide. Figure 21 shows examples of puzzles and their solutions, and Figure 22 shows the full alphabet. In each case, coloring the rectangles the same as the (black and white) clues reveals the letter in black.

These puzzles were designed by hand to have unique solutions, while extensively aided by our own brute-force computer search to verify solvability and uniqueness, in a larger team. We presented this font in a paper that proved NP-completeness of Tatamibari puzzles [ABD+20].

[10]https://erikdemaine.org/fonts/tatamibari/

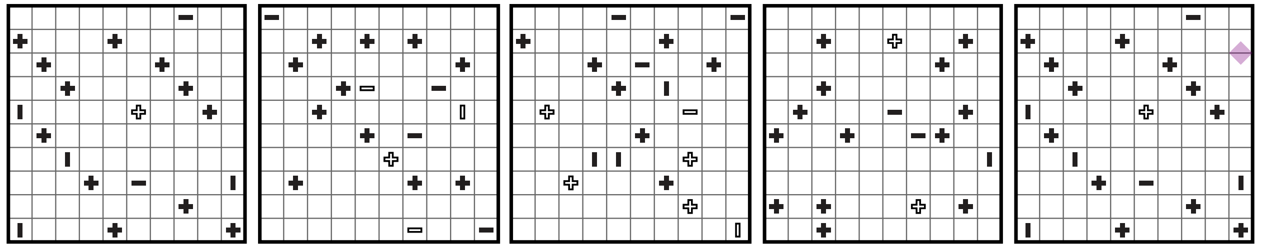

Figure 23: Can you draw one rectangle per clue so that plus clues are in squares, horizontal clues are in wider-than-square rectangles, and vertical clues are in taller-than-square rectangles? Or solve interactively on the web. The solution is in Figure 35.

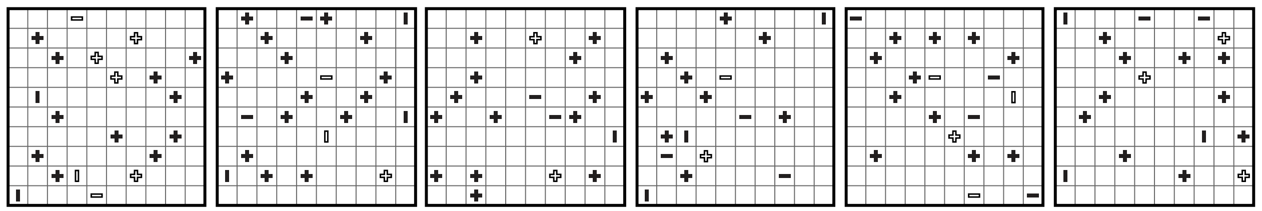

Figure 24: Can you draw one rectangle per clue so that plus clues are in squares, horizontal clues are in wider-than-square rectangles, and vertical clues are in taller-than-square rectangles? Or solve interactively on the web. The solution is in Figure 36.

References

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[IPS82] Alon Itai, Christos H. Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, November 1982.

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[YS03] Takayuki Yato and Takahiro Seta. Complexity and completeness of finding another solution and its application to puzzles. *IEICE Transactions on Fundamentals of Electronics, Communications, and Computer Sciences*, E86-A(5):1052–1060, 2003. Also IPSJ SIG Notes 2002-AL-87-2, 2002.

A Puzzle Solutions

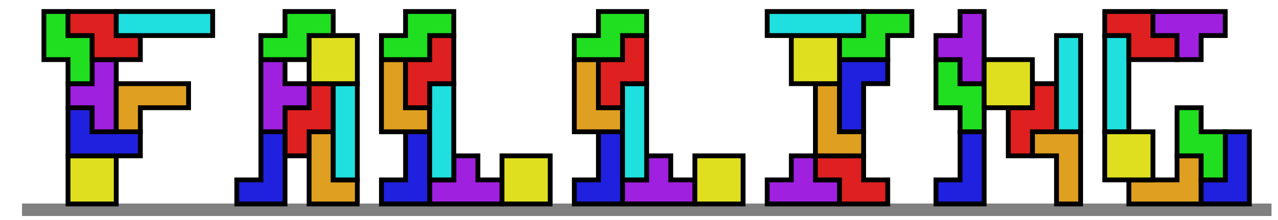


Figure 25: The self-referential message hidden in Figure 5.

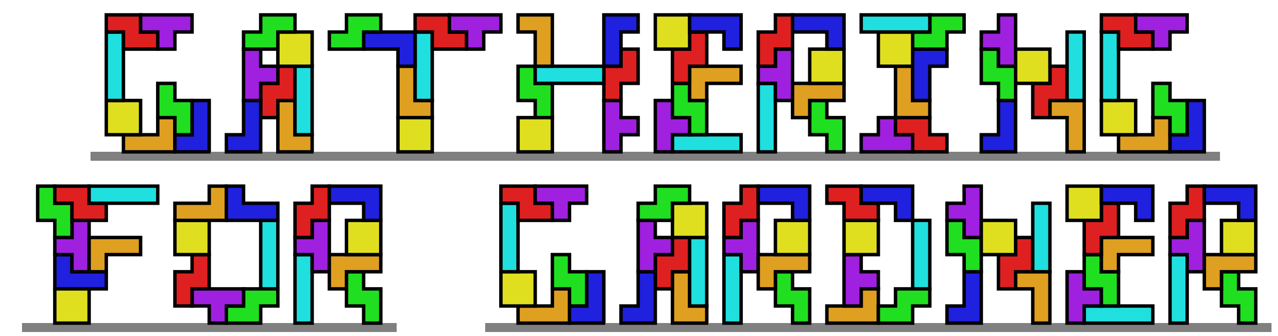


Figure 26: The message hidden in Figure 6.

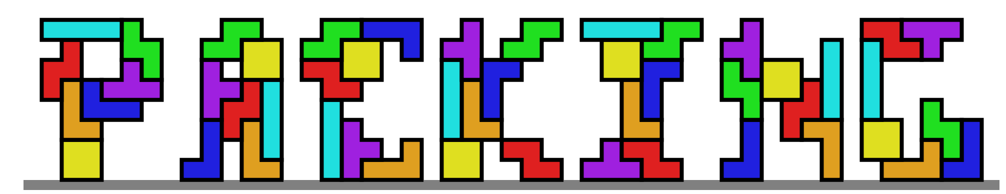


Figure 27: Tilings for the PACKING letter shapes in Figure 7.

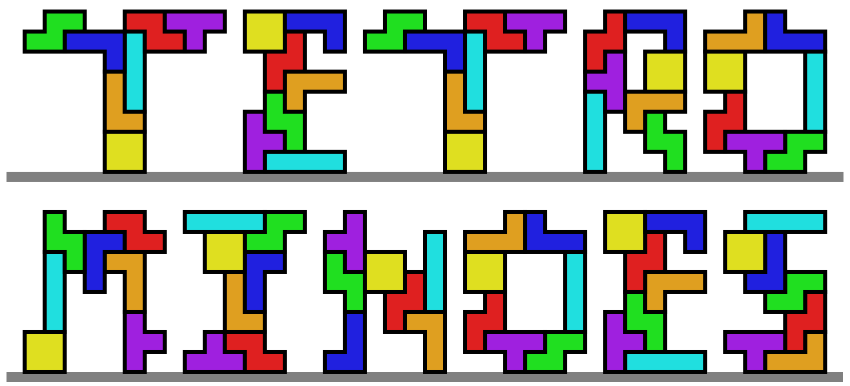


Figure 28: Tilings for the TETROMINOES letter shapes in Figure 8.

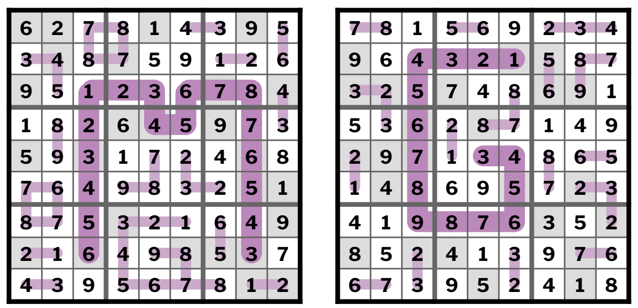


Figure 29: Solved Sudoku puzzles from Figure 11, in honor of Martin Gardner.

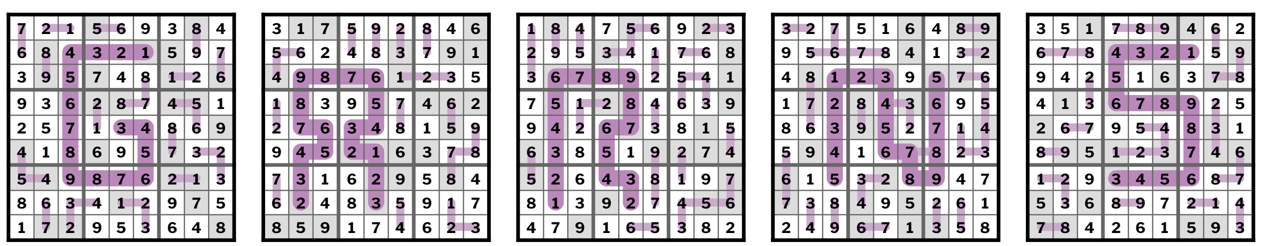


Figure 30: Solved Sudoku puzzles from Figure 12, in honor of Howard Garns who invented Sudoku puzzles (under the name "Number Place"), first published in May 1979.

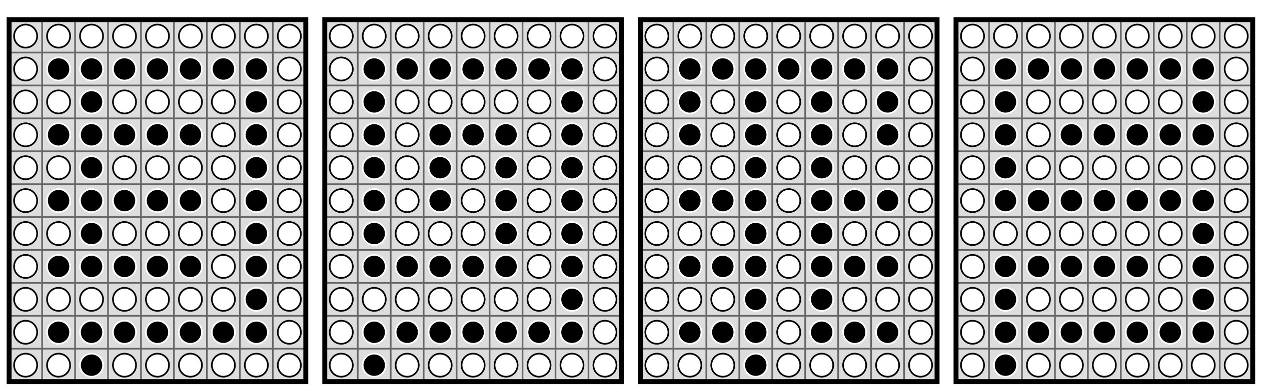


Figure 31: Solved Yin-Yang puzzles from Figure 14, revealing the self-referential message DOTS.

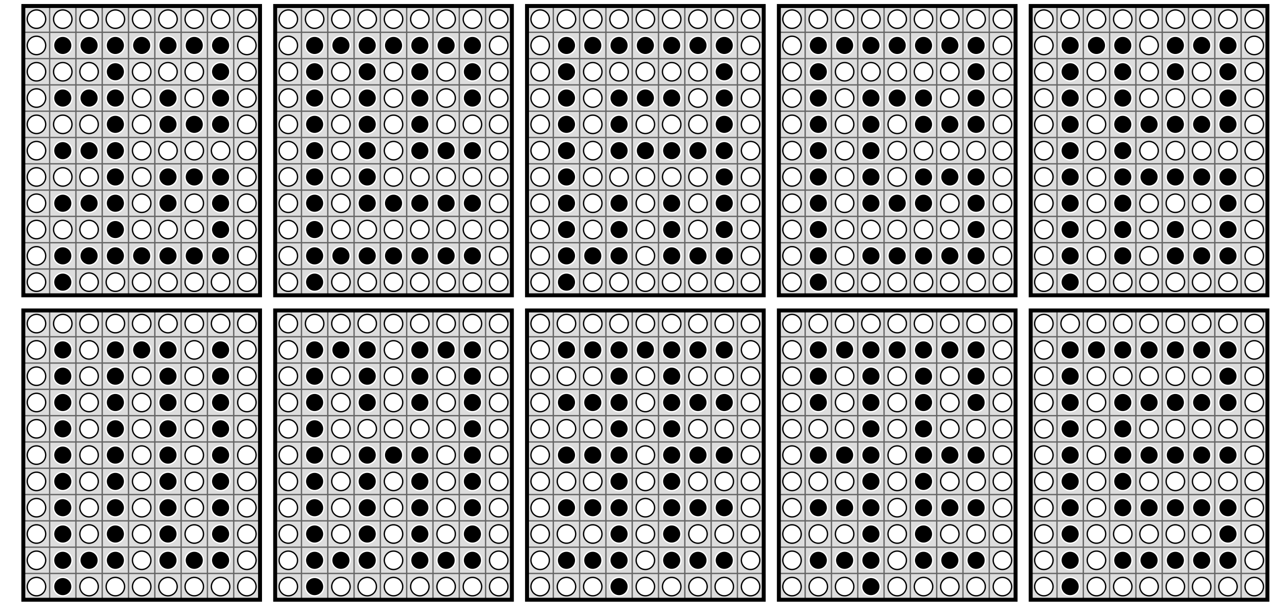


Figure 32: Solved Yin-Yang puzzles from Figure 16, revealing the self-referential message BLACK/WHITE.

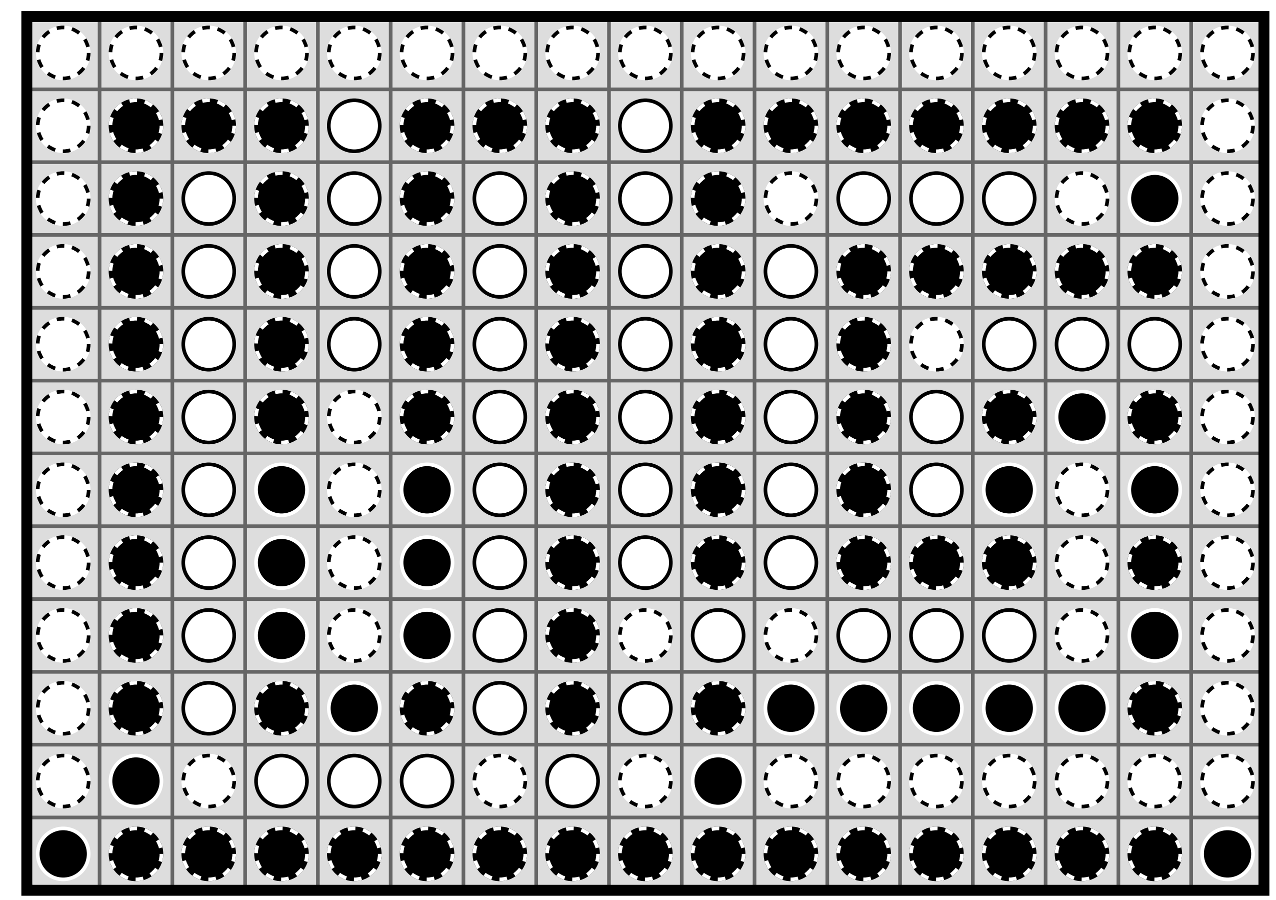


Figure 33: Solved Yin-Yang puzzle from Figure 14, in honor of Martin Gardner.

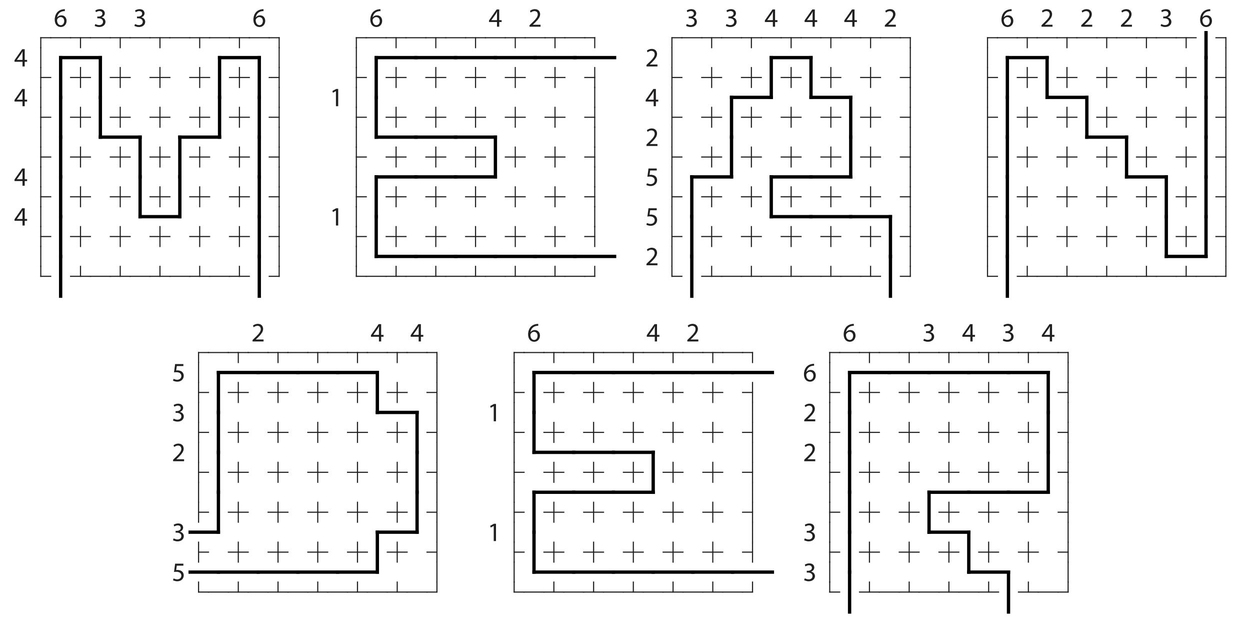


Figure 34: Solved path puzzles from Figure 19, revealing the self-referential message MEANDER.

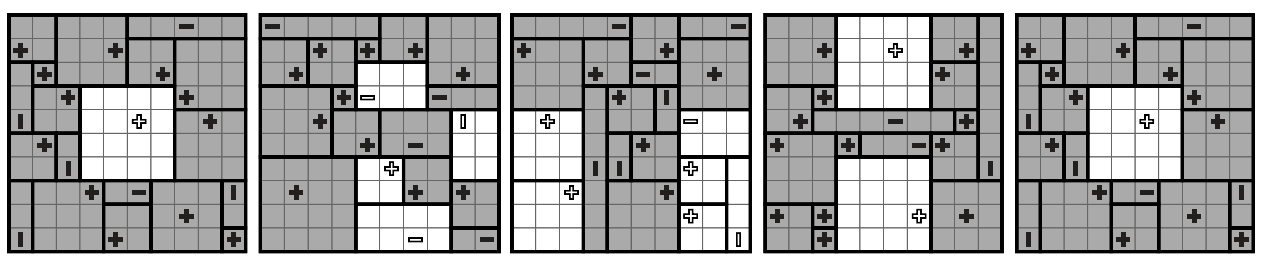


Figure 35: Solved Tatamibari puzzles from Figure 23, revealing the message ORTHO (referring to the orthogonal nature of rectangles).

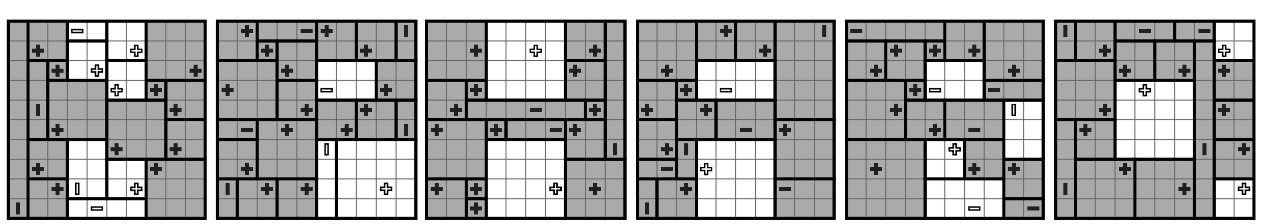


Figure 36: Solved Tatamibari puzzles from Figure 24, revealing the message NP HARD.

Multiplication by Superposition

By Doug Engel
Littleton, CO USA
March-11-2022

Multiplying polygons by superposition, Mps

For G4G14 this is an idea that can be used to devise various puzzles. The idea is to overlap two geometric figures and count the number of pieces produced if all lines are cut thru. The simplest case is two equal circles. We can overlap them in three different ways to get a^2' = 1, 2 and 3 pieces as shown in Fig 1. Call this multiplication Mps. These counts are the complete solution for this Mps. The primed expression, c^2', indicates Mps here.

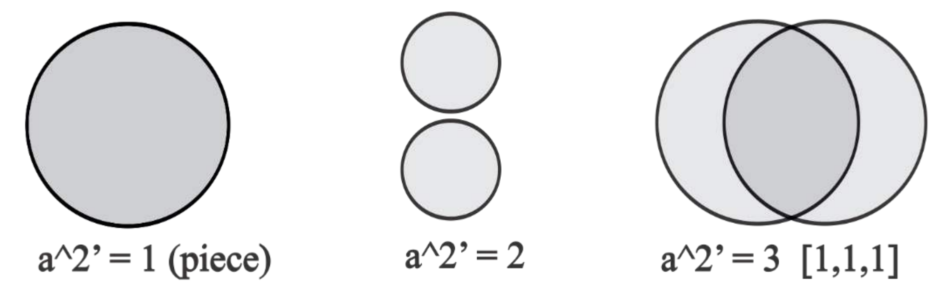


Figure 1

Figure 2 shows Mps solutions for 2 equal triangles a with a^2' having 7 solutions, 1 thru 7 pieces.

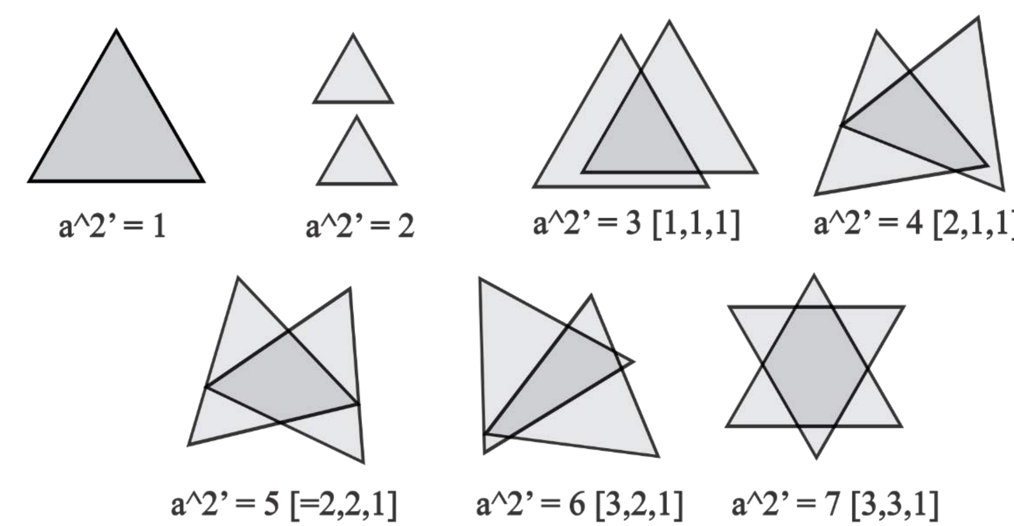


Figure 2

Continuing the sequence with a square we get 9 Mps solutions in Figure 3, and 11 solutions with a pentagon in Figure 4. The arrows in the 10 piece solution show where tiny pieces are located

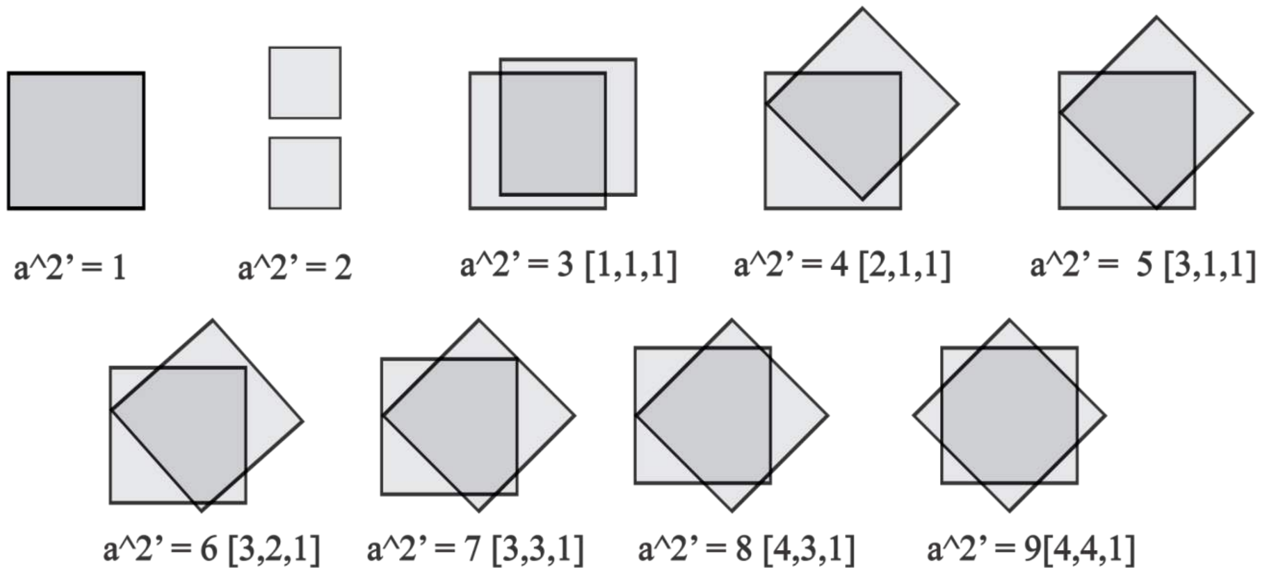


Figure 3

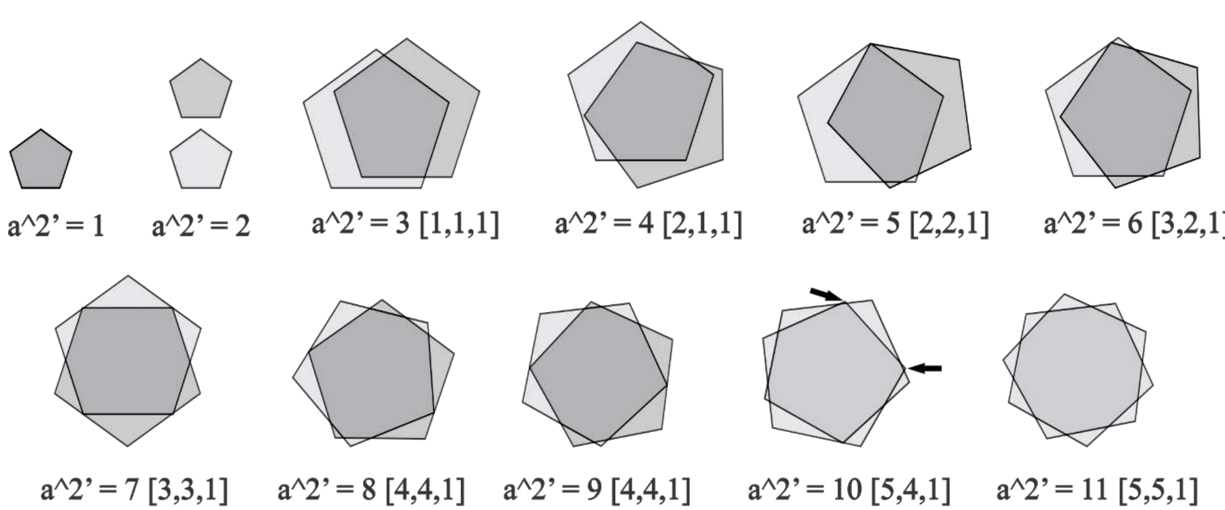


Figure 4

The number of solutions shown here for a regular polygon with n sides is 2n+1 for Mps. This has not been proven but is a conjecture based on the three solutions, triangle, square, pentagon. This could easily be refuted if the hexagon does not permit some solution less than 13.

If the a^2’ total solutions = 1 thru n+1 pieces continues for regular polygons the solutions will have increasing numbers of smaller pieces as n increases with a few large pieces as seen in the figures.

As n gets really large but still finite the polygons get more and more circular yet the number of solutions are n+1 ‘large’. At infinity we have a circle with only 3 solutions. Perhaps instead of solutions increasing they start to decrease as a ratio of s/n.

Spin Multiplication by superposition

Spin Multiplication of squares by superposition Sms

A square can be overlapped on top of itself and rotated 45 degrees to get 9 pieces when all lines are cut thru as shown in Figure 5. We overlap this product on itself and rotate it 22.5 degrees to get the next product having 49 pieces and then 11.25 degrees to get 225 pieces.

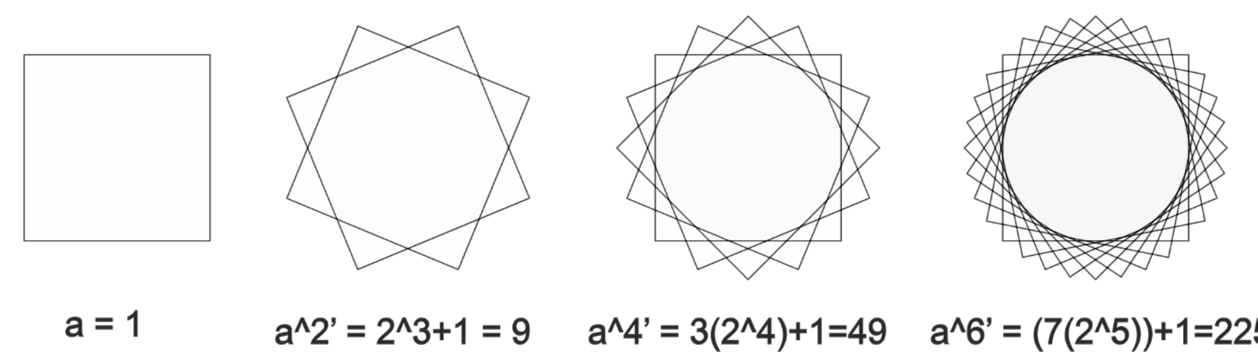
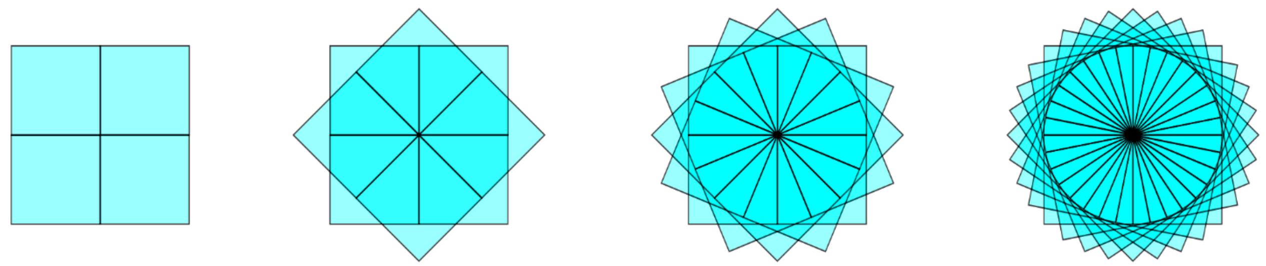


Figure 5

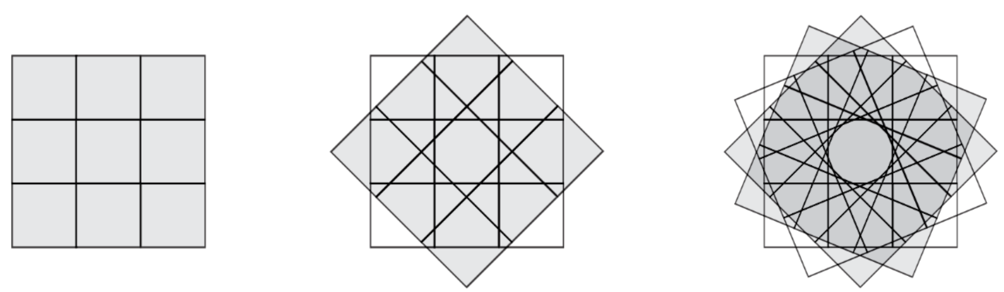
Overlapping a 2x2 grid gives the exponential sequence shown in Figure 6.



a^1’ = 4 a^2’ = 16 a^3’ = 64 a^4’ = 256

Figure 6

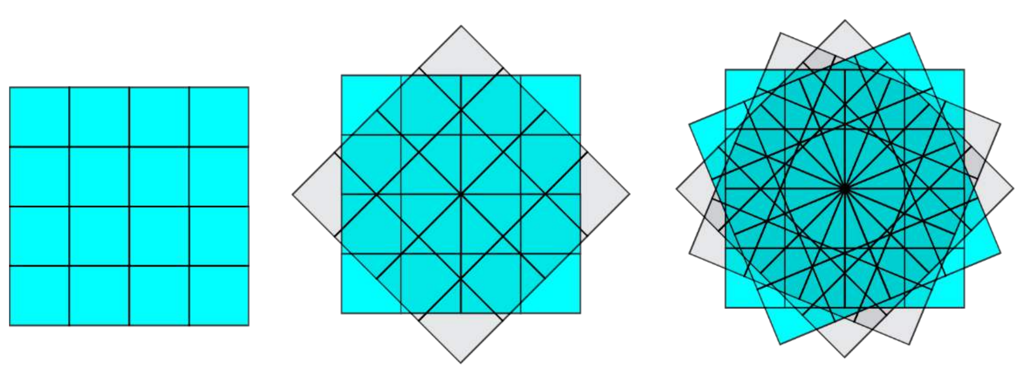
Odd order square grids have a slewed exponential sequence of pieces as seen in Figure 7 with a 3x3 grid.



a^1’ = 2^3 +1 = 9 a^2’ =5(2^3)+1 =41 a^3’ =12(2^4)+1 =193

Figure 7

Figure 8 with a 4x4 grid being even order returns a simple exponential sequence.

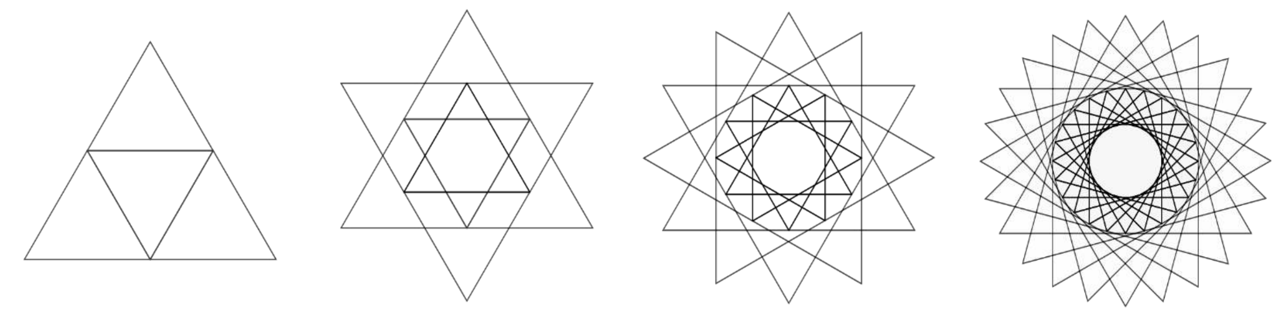


a^1’ = 4^2’ =16 a^2’ = 4^3=64 a^3 = 4^4’ =256

Figure 8

This regularity continues for all sorts of regular grids under this kind of binary symmetrical spin multiplication.

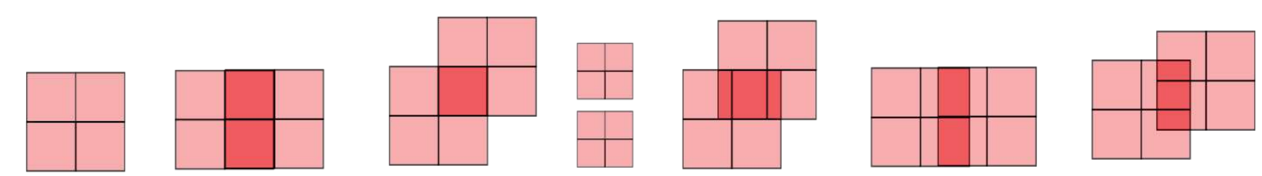
a^1’ = 1 a^2’ = 1x6+1 = 7 a^4’ = 3x12+1 = 37 a^6’ = 7X24+1 = 169



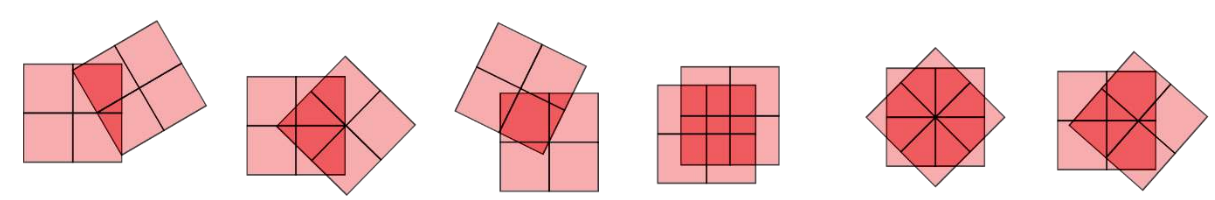
a^1’ = 4 a^2’ = 3x6 + 1 = 19 a^4’ = 7x12 + 1 = 85 a^6’ = 15x24 + 1 = 360

Figure 9

Things get more complicated if we allow all three operations, translation rotation and spin. This was attempted in Figure 10 with results as shown.



a = 4 6 7 8 9 10 11



12 13 14 15 16 17

Figure 10

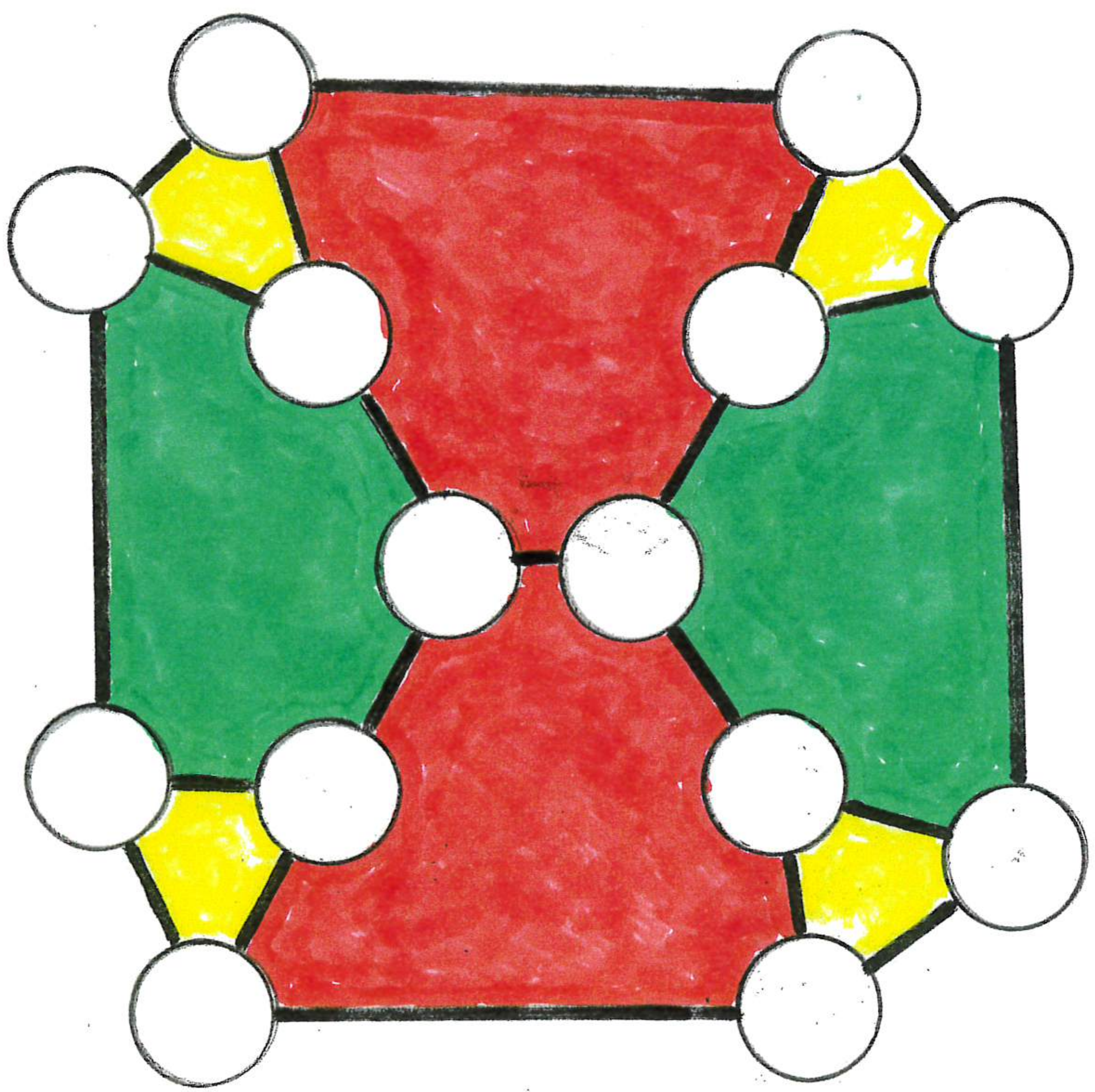
There is much more to explore

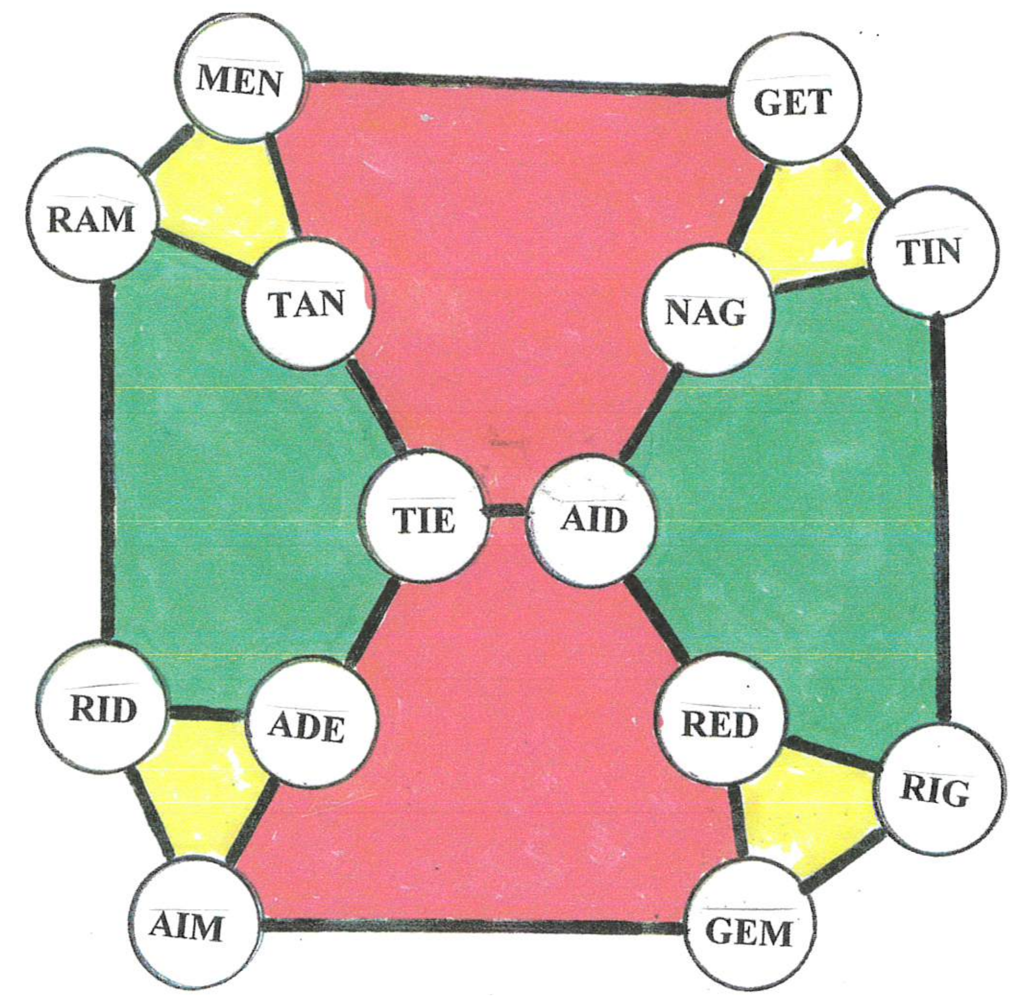
A 14 PUZZLE FOR G4G14

by Todd W. Estroff and Jeremiah Farrell

We have supplied 14 words using only the letters of MARTIN GARDNER to use in a puzzle. The words are ADE, AID, AIM, GEM, GET, MEN, NAG, RAM, RED, RID, RIG, TAN, TIE, TIN. Notice that each vowel is used 6 times and each consonant 4 times.

The Puzzle: Place the words on the nodes so that every two nodes are connected with a common letter.

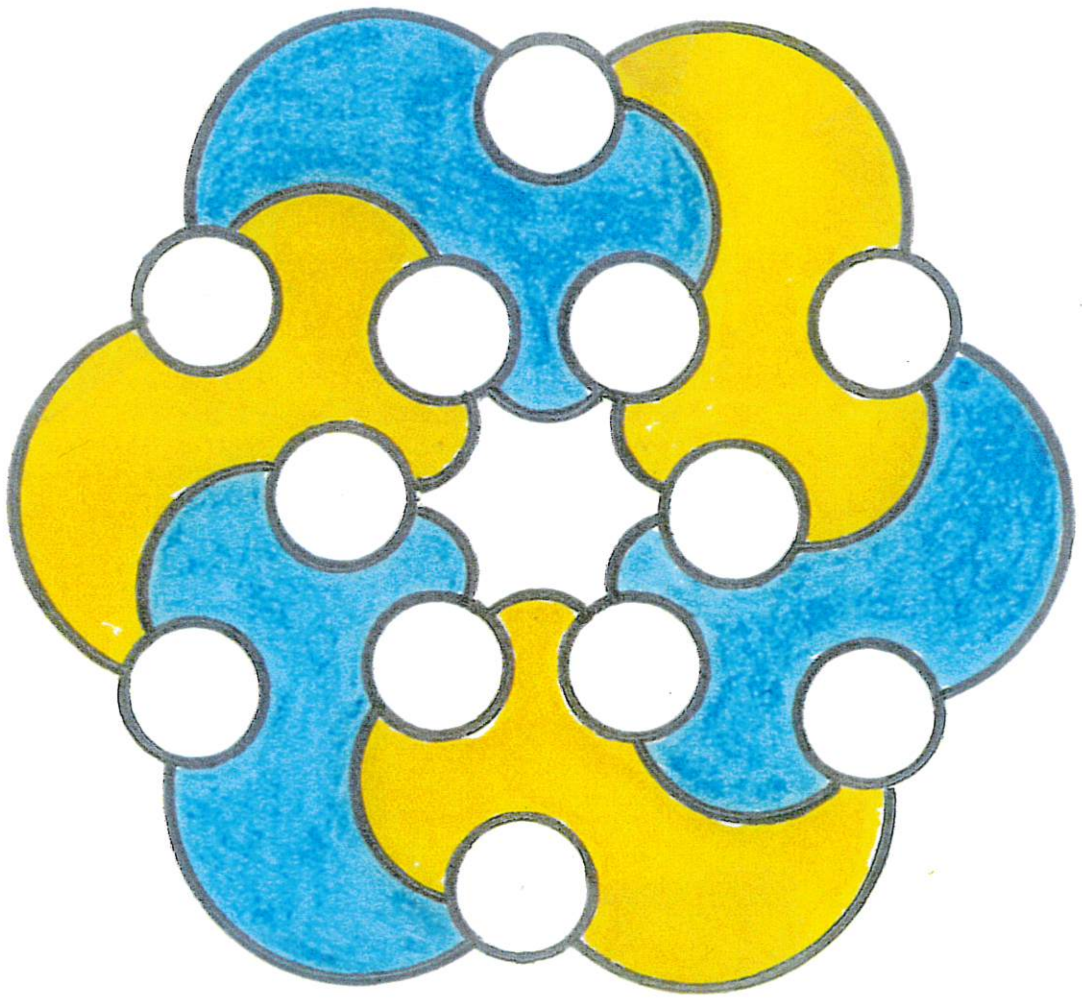




KING OF HEARTS

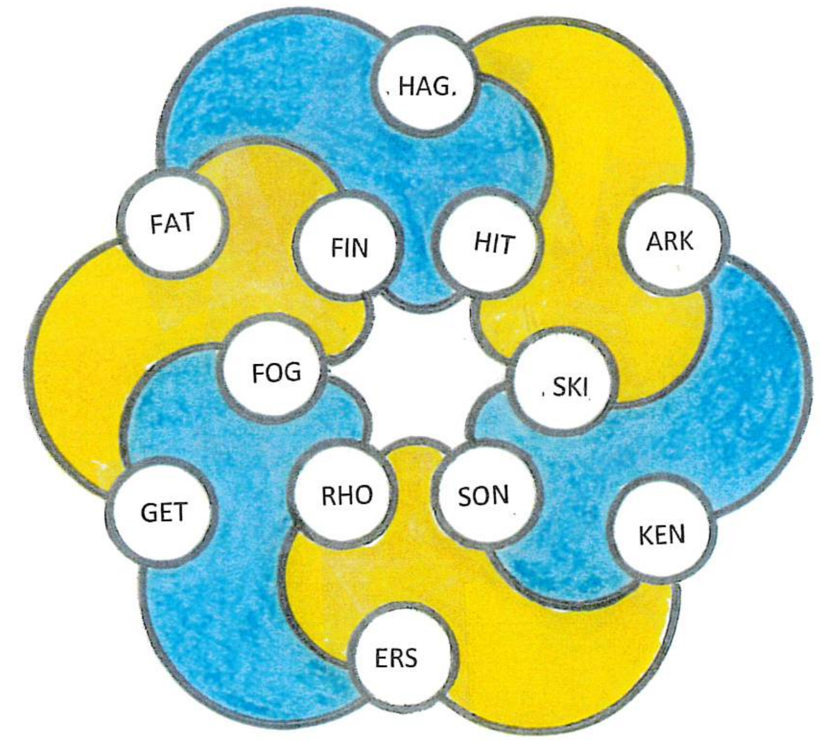
By Jeremiah and Karen Farrell

This is the Schlegal diagram of a hexagonal prism, a solid with a hexagon as base and top and with quadrilateral sides.



As a puzzle place the following 12 words on the nodes so that every two connected nodes have a letter in common. The words are formed from the letters of "KING OF HEARTS" used three times each.

ARK, ERS, FAT, FIN, FOG, GET, HAG, HIT, KEN, RHO, SKI, SON



Puzzle Books

by Rik van Grol

Introduction

In the spring of 2020, when the first COVID19 lockdown in the Netherlands was lifted and shopping was allowed again, I went to Amsterdam to visit antique shops to hunt for puzzles. I found several puzzles I knew were there and considered too pricy/not interesting enough, but I also stumbled onto two small puzzle books. One was incomplete, but was easily restored, the other was fine. More importantly, they have a locking mechanism I had not seen before. I have several puzzle books and for me this made me cross the point to allow me to say that I own a collection of puzzle books. So, I thought this was also a good moment to say something about them.

So, puzzle books are the subject of this paper. If, for some reason, you were thinking about collections of books about puzzles, then I have to disappoint you. In this paper I am presenting a collection of mechanical puzzles that look like books.

I define a puzzle book as a secret opening box in the shape of a book. Puzzle books have the size of a regular book, but that is not mandatory. Most puzzle books could not really hide in a bookcase; they would stand out. There are some “book-safes” that can actually hide very well and are intended to do so, but those are not puzzles. That puzzle books stand out is mostly due to the fact that they generally are made of wood.

I intended to write this article for the postponed G4G14 in April 2021, but it was postponed two more times. In the meantime, I have been collecting more puzzle books. And on the way to G4G14 I visited the Lilly Library in Bloomington (IN) to look at the puzzle books in the Slocum Collection. In this article I am presenting 14 puzzle books in my possession that differ by their locking mechanism and/or their country of origin. Figure 1 shows the row of puzzle books presented. One puzzle is not included here because it would not fit (number 3). That —knowing one is not incuded— the number is bigger than 14 is because I have several copies of some puzzle books. Those copies look different but have the same origin/locking mechanism.

Figure 1. Puzzle books presented in this paper, showing the range of sizes

Puzzle books from Hungary

1

If you visit Budapest, the capital of Hungary, and you wander into the Great Market Hall it is hard not to stumble on the first puzzle book presented here, see Figure 2. Also in souvenir shops throughout Budapest it is impossible for a puzzle collector not to notice these puzzles (unless you are fully focused on the other well-known puzzle from Budapest, a Sorrento-type puzzle box, which is even more dominantly available is all sorts and sizes). This puzzle book has a simple well-known opening, which is discussed in a book by Jack Botermans and Jerry Slocum [1]. The opening is in the spine of the book and can easily be recognized by a skew cut through the spine, see Figure 2. This lock is well-known, so I do not feel I give away anything.

Figure 2. Hungarian puzzle book and its locking mechanism (left & middle), and concealing the skew cut in the spine by smart decoration (right)

Note that these puzzle books are not really sold as puzzles, but more as souvenirs. You can get this puzzle in many different designs – some traditional, others beautifully decorated. In some puzzles the lock is better concealed then others. In one of the books the skew cut in the spine is concealed by a skew carving pattern on the spine, see Figure 2 right. This type of puzzle book was the first I purchased, several decades ago. At later visits to Budapest I purchased additional copies, see Figure 3. These puzzles have the size of a small pocket book (Height ~13 cm, Depth ~10 cm, Thickness ~3 cm).

Figure 3. Range of typical Hungarian puzzle books

2

During a recent visit to Budapest [2] I found a puzzle book with a slightly different locking mechanism. The lock is simpler, probably because the puzzle is smaller and the original locking mechanism would be too big. The size is odd and small (HDT 9×8×2.5 cm), see Figure 4.

3 During the same visit to Budapest I went to the Ecseri Street Market (a large permanent flea market) and I found an antique puzzle which I previously had only seen in a book by Jerry Slocum and Jack Botermans [3], see Figure 4. It is a stack of two wooden books on top of a base. There is a hidden compartment in each of the books, and a third compartment in the base. This antique puzzle is beautifully made with inlaid wooden decorations and finished with French polish. The base is 35×28 cm, the height is 12 cm.

Figure 4 Small Hungarian puzzle book (left) and puzzle-book-stack (right)

Puzzle books from Sri Lanka
I have never visited Sri Lanka, but I managed to get several of their puzzle books. It is easy to recognize puzzle books from Sri Lanka. They are hand-carved and depict one or more elephants on the front and generally a woman's face on the back, see figures 5 and 6.

Figure 5

4 The first type has one big compartment that opens at the front of the book (unlike the Hungarian book that opened at the spine). The locking mechanism is simple, see Figure 6 middle. I found two copies of this puzzle in second hand stores in the Netherlands.

5 The second type has four small compartments that all open from the front (Figure 6 right). The locking mechanism is the same as with the first type. This puzzle I purchased from Frans de Vreugd who visited Sri Lanka together with Peter Hajek [4] in 2012.

Figure 6. Books from Sri Lanka; simple (left/middle), more complex (right)

Antique puzzle books from eBay
With my interest in puzzle books I searched eBay for antique puzzle books. There is enough to be found, but it is not always easy to find samples that are reasonably priced and that you actually manage to purchase. Perseverance and some luck is required. Over the past few years I purchased three antique puzzle books.

6 The first is a puzzle book that, as a puzzle, resembles the Sri Lanka puzzles, in that the front of the book slides off. It has a nice inlay on the front, see Figure 7. This book, however, is not professionally made. It is cut roughly and the way it is manufactured is amateuristic. Due to some design choices its solution is not immediately obvious, so it is still a nice sample. The origin is probably England. The puzzle has pocket book size (HDT 16×12×3.5 cm).

Figure 7. A simple puzzle book with inlays, probably made by an amateur

7 The second puzzle book is a professionally made puzzle. It does not show any inlays or pretty decorations, but it perfectly conceals its solution, see Figure 8 left. Apart from its weight (light) it feels like a solid piece of wood. If you would hide something fairly heavy you might actually get away with it. This is a simple but very good puzzle book. The origin is probably England. The puzzle has pocket book size (HDT 16×12×3.5 cm).

8 The third puzzle book is a small Anglo-Indian puzzle book, made from teak wood with brass inlays, see Figure 8 right. As a puzzle it is extremely simple, if you know the solution… The solution is well-hidden; another example of a simple but good puzzle book. The size is that of a small pocket book (HDT 12×9×3.5 cm).

Figure 8. Two books with a well-hidden lock; English (left) and Indian (right)

Antique puzzle book from The Netherlands

9 This is the puzzle I found in Amsterdam in an antique shop with which I started this paper, see Figure 9 left. I got two identical copies. They are very small (HDT 8×5×2 cm). Interestingly the locking mechanism is one that I did not see before. As with most puzzle books, the locking mechanism is not difficult, but well-hidden.

Figure 9. The smallest puzzle book I own is from Amsterdam (left), the biggest from France (middle), and the thickest from Cyprus (right)

Antique puzzle book from France

10 Searching the internet I also found a puzzle book in France. This is the biggest I have seen so far (HDT 31.5×22.5×5 cm). The lock is simple and easily spotted. The book has a beautiful inlay of a village scene, see Figure 9 middle.

Puzzle book from Cyprus

11 Most puzzle books presented so far are old or traditional. The puzzle book I found online from Cyprus is a more modern puzzle book. The locking mechanism is a combination of known locking mechanisms. The puzzle is quite large and meant to store a bible (HDT 25.5×18.5×7 cm), see Figure 9 right.

Puzzle books from the International Puzzle Parties

Two of the puzzle boxes are Exchange puzzles for the International Puzzle Parties.

12 The first puzzle book is not meant to be a puzzle book. It is a storage box for a puzzle. It resembles a book, and is therefore included here. This puzzle is the Exchange puzzle from George Bell in 2015 at IPP35 in Ottawa, see Figure 10.

Figure 9. IPP35 Exchange puzzle from George Bell, closed and open

13 The second puzzle book is more a puzzle than a puzzle book. It looks like a book but is a sequential discovery box. After opening the book, which requires the largest number of steps of all presented puzzle books, there is a small storage area which contains a small puzzle. The book is designed by Louis Coolen and Adin Townsend, and exchanged at IPP34 in London in 2014 by Allard Walker, see Figure 10.

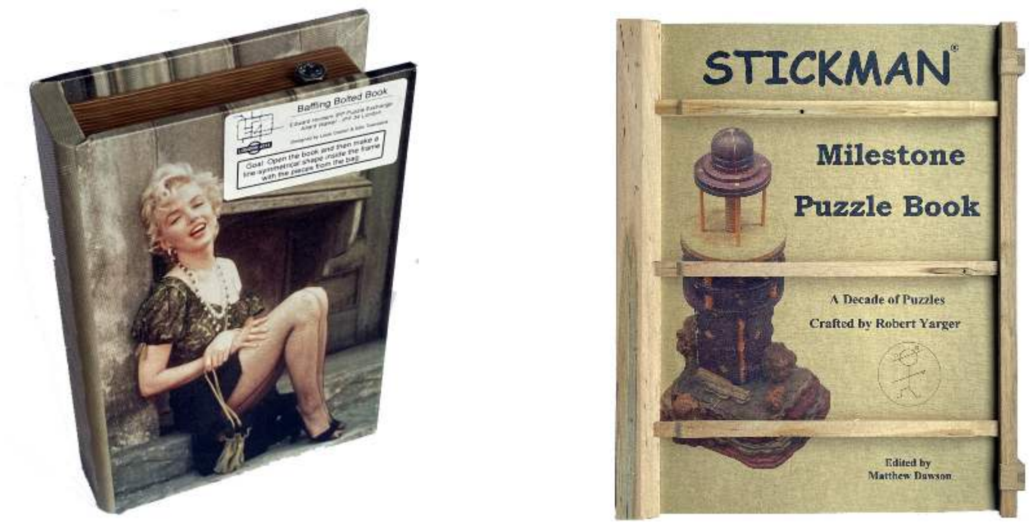


Figure 10. IPP34 Exchange puzzle from Allard Walker (left), book in puzzle lock from Robert Yarger (right)

Puzzle book from USA

14 Puzzle book number 14 is not a puzzle box that looks like a book, but an actual book that is locked by a puzzle, see Figure 10. The book is a book by Robert Yarger about his puzzles [5], STICKMAN Milestone Puzzle Book.

Design features of a good puzzle book

A good puzzle book is a good puzzle. The locking mechanism may be simple, as long as the lock is well-hidden. The next requirement would be to look like a real book. This would mean that the pages should look like pages as much as possible. The ultimate puzzle book might have a leather or cloth spine or even a complete cover so that it could be hidden on a bookshelf. A simple solution in order to hide a puzzle book amongst normal books might be to give the book a dust jacket.

There is much more out there

In this paper 14 puzzle books were shown. But there is much more out there. Most presented puzzle books are traditional and/or antique. I visited the Slocum Collection to see if I had missed obvious examples of puzzle books. This is not the case. The most important puzzle book not presented is the puzzle book from Akio Kamei. This puzzle book is well made, but quite traditional. It has a simple, not particularly well-hidden solution.

In recent times puzzle books with new and more difficult locking mechanisms have been developed. Some new puzzle books can be found on https:\\www.Etsy.com, and in books on puzzles.

Puzzle books in literature

To find puzzle books in literature is not easy. Before recently, at most one or two puzzle books would be shown in only a handful of books, mostly by Jerry Slocum and Jack Botermans, see the references. Some puzzle books are documented in over 30 years of the Dutch puzzle magazine CFF. Recently a book about puzzle boxes has been published by Peter Hajek [6]. In this book quite a few puzzle books are discussed and presented. Also some recent puzzle books are documented in this book. It was published while this article was being written: amongst other puzzle boxes it shows more puzzle books than presented in this article and previously in other books.

For publications from the Dutch Cube Club, Cubism for Fun (CFF), see the website: http://cff.helm.lu.

References

[1] Jerry Slocum and Jack Botermans, *Puzzles Old and New*, ISBN I-85336-018-X, 1986, p. 53.

[2] Rik van Grol, *Hunting for Puzzles in Eastern Europe*, CFF 116 (November 2021), pp 12-15.

[3] Jerry Slocum and Jack Botermans, *Het Ultieme Puzzelboek*, ISBN 978 90 5897 720 5, 2007, p. 48.

[4] Peter Hajek, *Sri Lanka Carvings with Secret Compartment*, CFF 89 (November 2012), pp 13-15.

[5] Robert Yarger, *Stickman Milestone Puzzle Book – A Decade of Puzzles*, private publication.

[6] Peter Hajek, *ENTER IF YOU CAN - The art of puzzle boxes*, ISBN 978-1-5272-8215-5, 2021.

[7] Rik van Grol, *Books and Magazines*, CFF 117 (March 2022), pp 30-31.

Sherclock Bird Watching

Jim Guinn

Hi Folks,

For about the past four years I have been writing a math/puzzle story for the bi-monthly publication of the National Association of Watch and Clock Collectors (NAWCC). The puzzles always involve my two protagonists Mr. Sherclock Holmes and his friend Dr. John Watchson (I am quite proud of coming up with those!). Here is one of the puzzles (with NAWCC changed to G4G14). The answer is below. I hope you enjoy it. ~Jim Guinn

From *Mart & Highlights*, published by and used with permission from the National Association of Watch & Clock Collectors, Inc.

The Adventures of Sherclock Holmes – Bird Watching

Consulting Time Expert Sherclock Holmes entered their London apartment to find his good friend Dr. John Watchson staring at his new Cuckoo Clock. “Taken up bird watching, I see, Watchson.” “Very funny, Sherclock. Actually I’m puzzled about something”, said Watchson. “Well, that’s not unusual. What is it?” said Sherclock. “I’ve noticed that of the two weights on this cuckoo clock, the left one drives the time hands, the right one drives the cuckoo. The chains themselves are identical. The left one descends one link every two minutes, while the right one descends four links for every cuckoo. It starts to cuckoo on the hour one cuckoo per hour, and on the half-hour just one cuckoo. I started the weights at exactly the same height at twelve o’clock just before it started to cuckoo. I’m wondering if the weights will ever be at exactly the same height again.” “Well,” said Sherclock, “you can stand there and watch it for the next twelve hours, or I’m sure someone from G4G14 can help you with it. I’m going to take a nap and please don’t let your cuckoo fiddling wake me up!”

Can you help Dr. Watchson figure out this Cuckoo Puzzle? That is, can you determine the time, or times, when the weights will be at the same height? To determine the exact time, you would need to know how the time weight descends as the clock ticks, and how quickly the cuckoo descends with each cuckoo. For this puzzle, finding the time(s) to the closest minute is fine.

Answer:

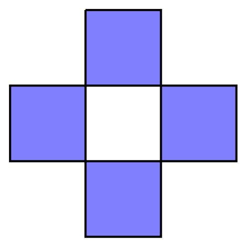
Amazingly, (according to my calculations) the weights will be the same height at six different times (per twelve hour period)! The first will be after the weights have each dropped 60 links, at 2:00 o’clock just before the cuckoo cuckoos. The second is at 68 links, at 2:16 o’clock, the third at 300 links at 10:00 o’clock after the cuckoo has cuckooed 8 of its 10 cuckoos, the fourth at 308 links at 10:16 o’clock, the fifth at 330 links at 11:00 o’clock after the cuckoo has cuckooed 4 ½ of its 11 cuckoos, and then the sixth and last, at 360 links, which is where the weights started out together at 12:00 o’clock just before the cuckoo cuckoos!

Stamping a Checker Board

Rachel Hardeman Morrill

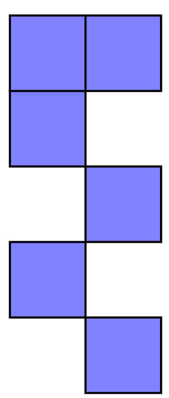
July 1, 2022

Suppose you have an $n \times n$ checker board with n odd. A *stamp with m blocks* is a configuration of m blocks that we do not require to be adjacent. For example, the following is a stamp with 4 blocks.

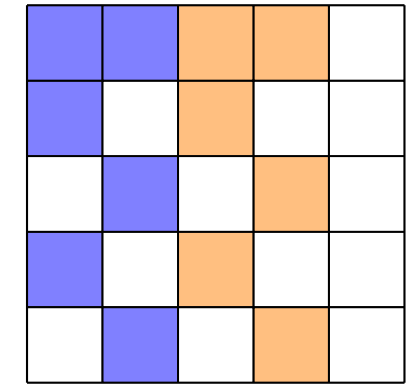


Can you use a stamp with $n+1$ blocks $n-1$ times to cover all but one block of your $n \times n$ checker board? The answer is yes! There is a way to cover the board such that the missing block is in the corner. Since n is odd, there exists an integer k such that $n = 2k+1$. If k is even, there is a way to cover the board so that the missed block is in the center. I will describe both ways of stamping the checker board in this brief article.

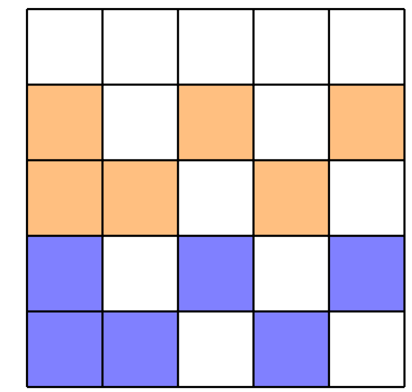
In order to cover all but one block of the board, the stamp with $n+1$ block must not hang off the board for any of the $n-1$ uses. For the covering that misses a block in a corner, consider a stamp on a $2 \times n$ grid where there is a block in the first column of the first row, a block in the first column of every even row, and a block in the second column of every odd row. Here is an example of the stamp when $n = 5$ to demonstrate.



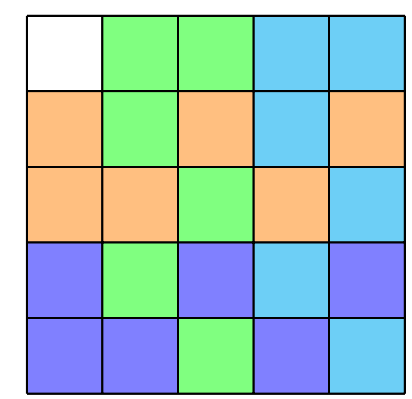
Since there is one block in rows 2 to $n-1$ and two blocks on the first row, there are a total of $n+1$ blocks in the stamp. Starting at the top left corner of the board, use the stamp k times as you move to the right. The result should have all but the last block of the first row of the board covered and a checkered covering of rows 2 to n and columns 1 to $n-1$. This is depicted on a 5×5 board in the image below.



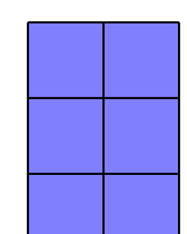
Now turn the board a quarter turn counterclockwise. The top row should now have no blocks covered.



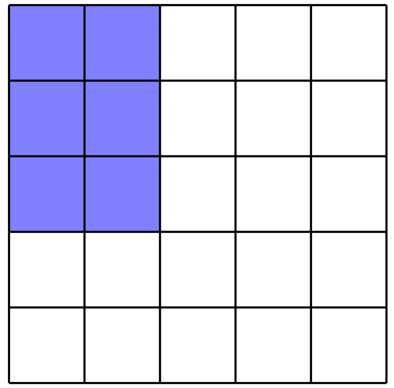
Starting at the second column, use the stamp k times as you moved to the right. This will fill in the checkered part of the board from row 2 to n and column 2 to n and all but the first block of the first row. The full stamp covering is demonstrated on a 5×5 board below.



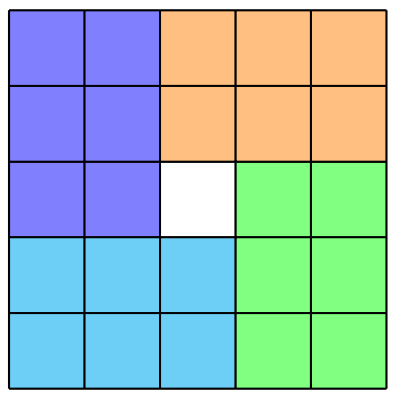
For the covering that misses the center block, suppose that $n = 2k+1$ with k even. Consider a stamp that is 2 blocks by $k+1$ blocks. Then the stamp has $2(k+1) = 2k+2 = n+1$ blocks. The stamp for a 5×5 grid is shown below.



Starting at the top left corner of the board, stamp the board $k/2$ times. Since k is even, this is an integer. The result is illustrated below for a 5×5 board.



Now turn the board a quarter turn counterclockwise. Starting at the top left corner of the board, stamp the board $k/2$ times. Repeat this action 2 more times, and the board will be covered with only the center block not stamped. The full stamp covering is demonstrated on a 5×5 board below.

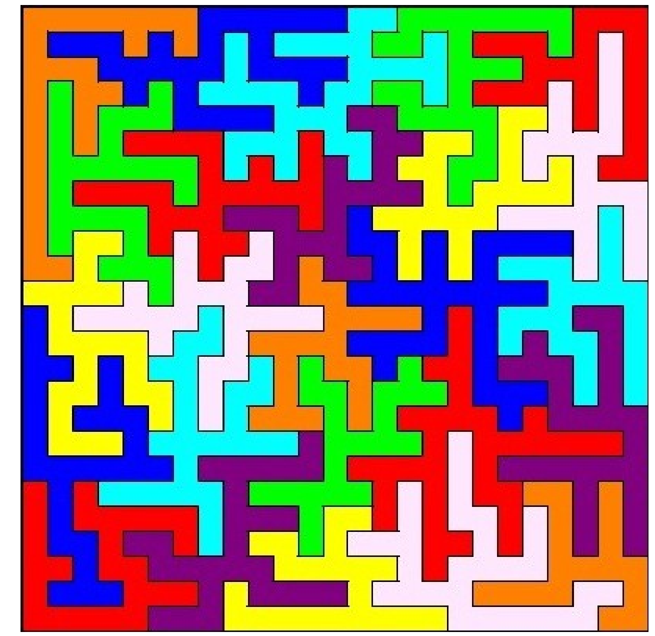


This problem was a generalization of a question on the *Fall 2019 Tournament of Towns*.

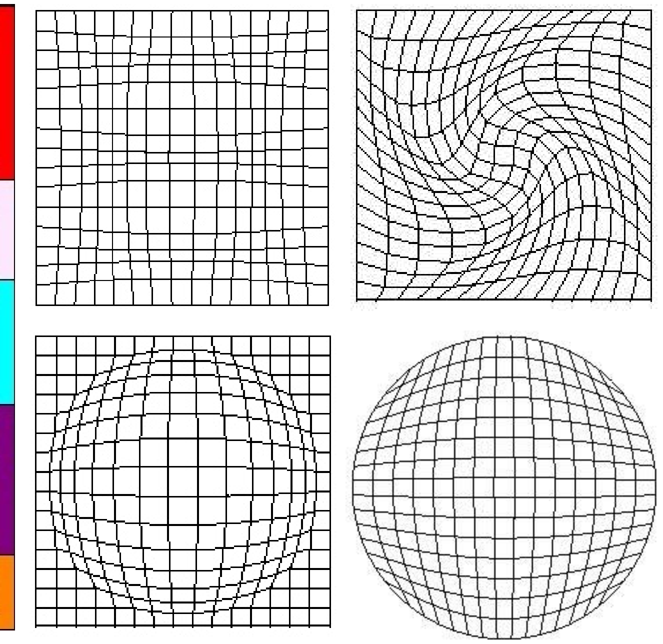
Warped-Grid Jig-Saw Puzzles

George Hart

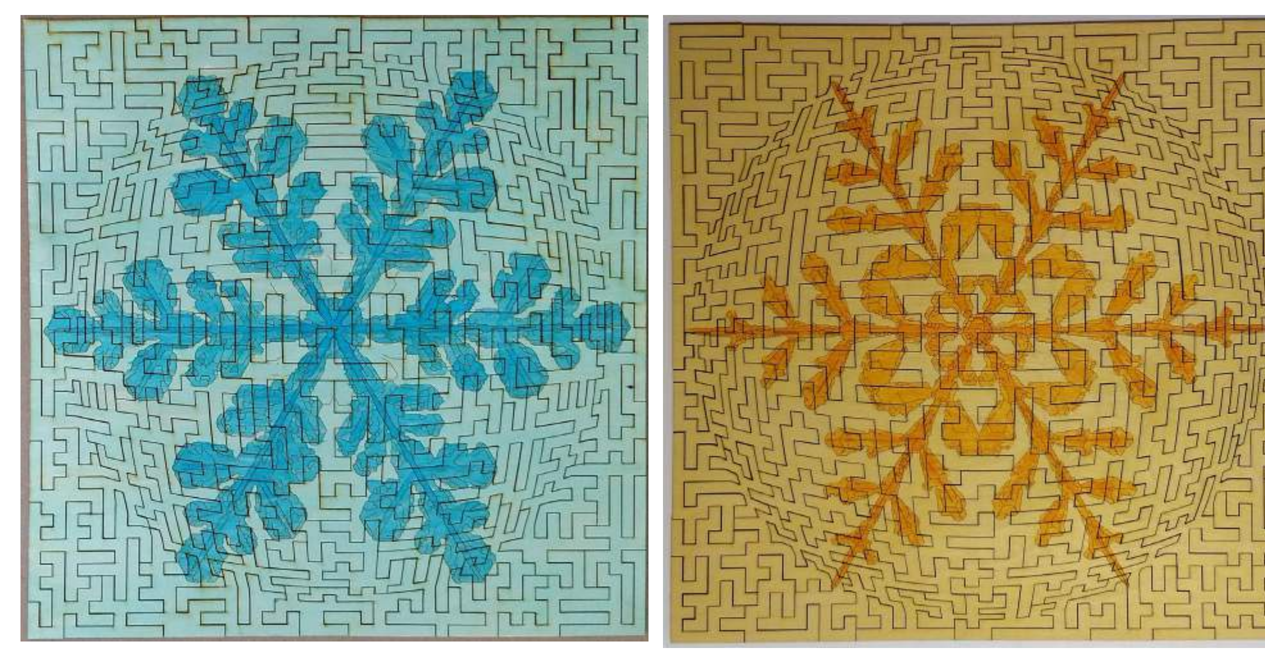
With a laser-cutter, it is easy to make one-of-a-kind jigsaw puzzles. The question then is what pattern of cuts to use. Here is a gallery of some experiments I have been making in which the cut pattern is based on a warped grid. A pseudorandom grid-based dissection of an underlying rectangular array is distorted to give engaging visual effects.



Example pseudorandomly generated 25-piece puzzle based on square grid, before warping the grid. Such designs can be warped in many ways.



Examples of warped 15×15 grids based on various easy-to-program transformations of the unit square: a) slight curve, b) swirl, c) bubble, and d) circle



100-piece laser-cut wood puzzles (with handpainted snowflake images) based on "bubble" warp, 11.75 inch square. Left example contains some 2×2 and larger blocks of cells, e.g., top-left corner, which I later decided should be avoided. Right example avoids any such blocks. (The "bubble" warp is inspired by op-art paintings by Victor Vasarely. The snowflake designs are based on photos by Kenneth Libbrecht.)

80-Piece puzzle with simpler part shapes, using a slight curve warping of the grid, 8×10 inches. (Etched image is Seven Ballerinas by Picasso.)

80-piece puzzle using swirl pattern. 8×10 printed photo was glued to wood before laser-cutting. (Image is Ocean Park #24 by Richard Diebenkorn.)

100-piece "tree rings" puzzle based on circular warping of grid. Laser-etched circles are scaled in geometric proportion.

100-piece puzzle using contraction transformation of grid. Image is hand-painted based on Ernst Haeckel drawing of Siphonophorae from Artforms in Nature.

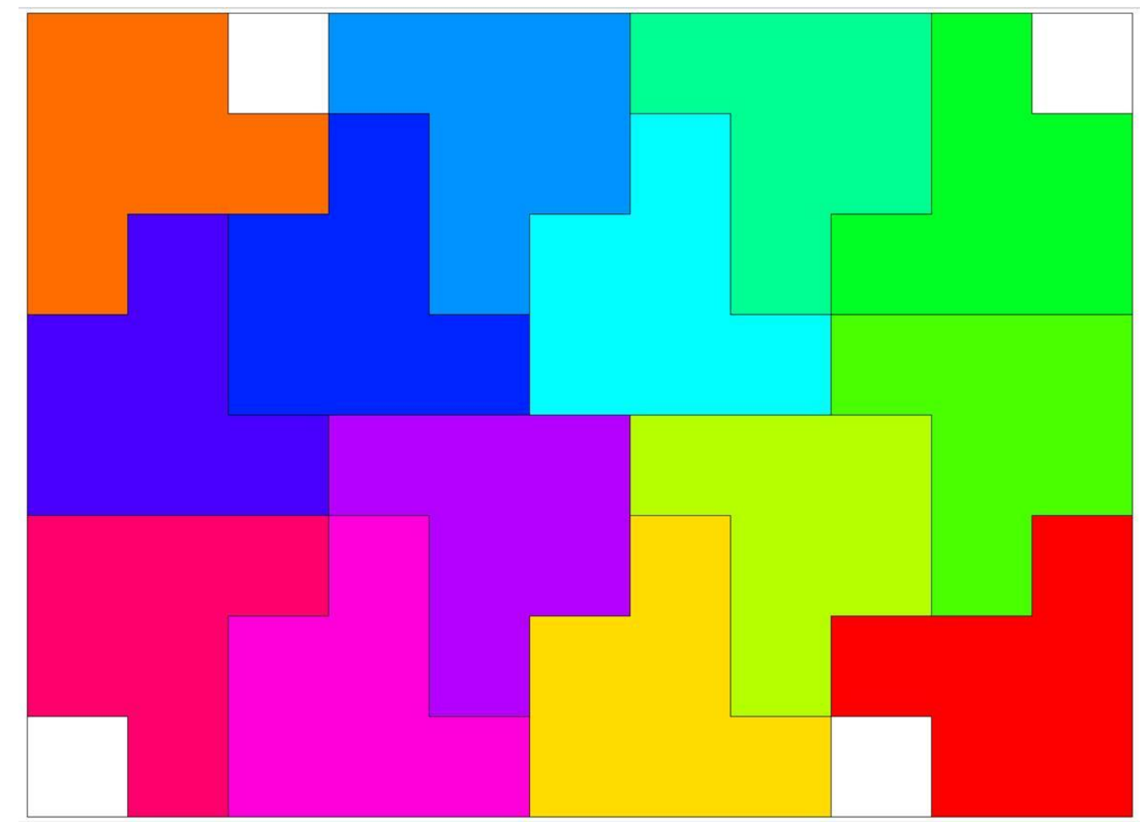
This family of puzzles can be adapted to any desired level of complexity by choosing the resolution of the underlying grid. An infinite variety of warping transformations can be coded up using straightforward mathematical techniques to change the visual effect. Test-solvers report that these puzzles provide a fun solving experience. My G4G-14 exchange item is a small 25-piece puzzle with a bubble transformation. For more information, see: `http://georgehart.com/jigsaw`

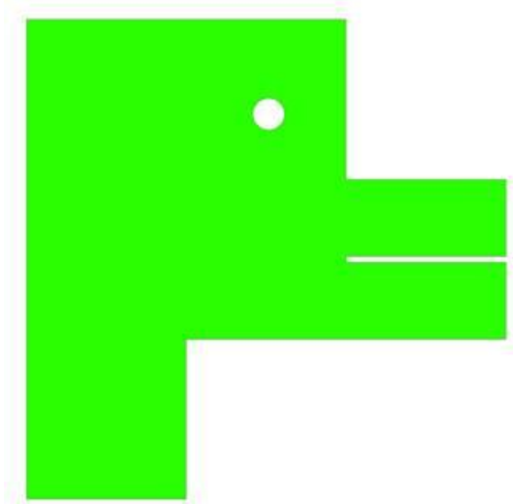
14 Rabbiducks

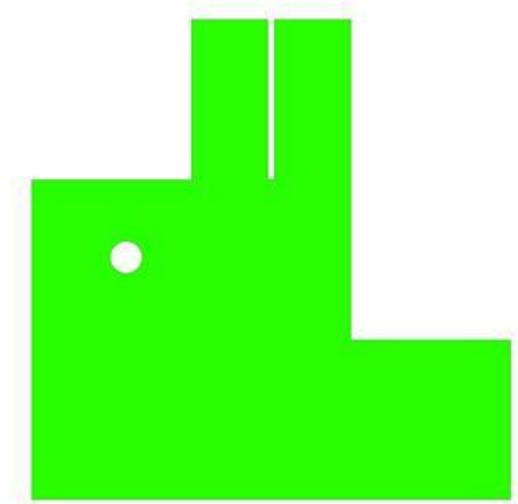
Haym Hirsh

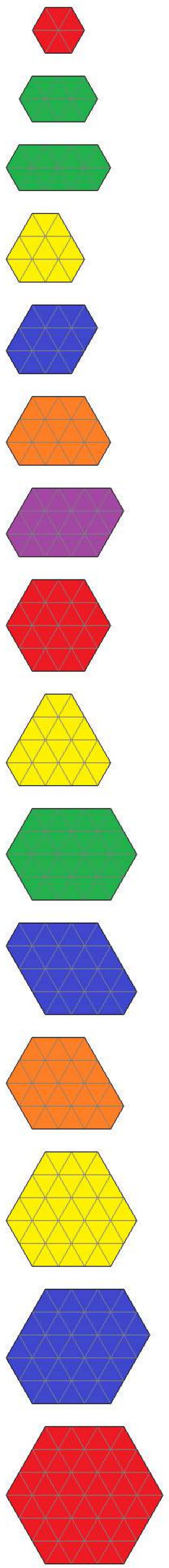
Explanation: The puzzle is to fit 14 "rabbiduck" polyomino pieces into an 8x11 rectangle. (Gardner wrote about polyominoes in multiple columns. The specific polyomino pieces used are inspired by the rabbit/duck illusion that Gardner called the Rabbitduck in his autobiography.)

14 "Rabbiduck" Pieces that look (crudely) like a rabbit or a duck depending on orientation:









Hex-Pave

by Carl Hoff (carl.n.hoff@gmail.com)

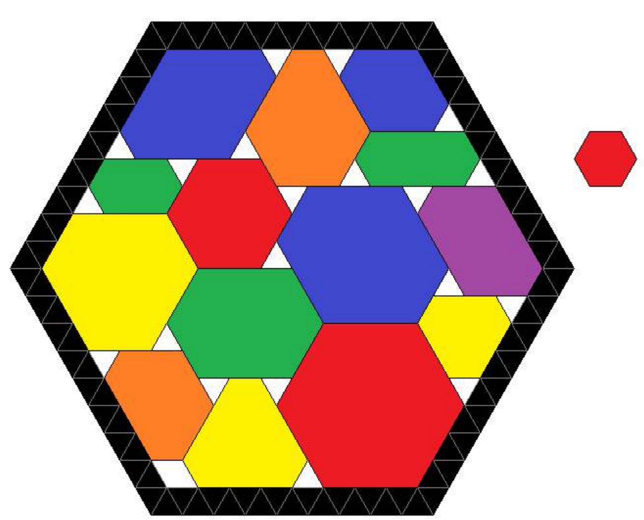
There are 15 integer sided equiangular convex hexagons with edge lengths of 1, 2, and 3 units (seen to the left). The main objective is to pack all 15 pieces into a size 8 regular hexagon (the black boarder shown below). This puzzle has 19 solutions.

Some auxiliary objectives can also be considered:

(1) Pack the 14 pieces left after the smallest piece is removed.
(3051 solutions)
  (a) and leave only 10 voids, the minimum possible, within the frame.
  (1 solution)
  (b) and leave 21 voids, the maximum possible, within the frame.
  (12 solutions if voids can touch at a corner, 3 solutions otherwise)
  (c) with no like colors sharing an edge.
  (75 solutions)
  (d) with no like colors touching, not even at a corner.
  (43 solutions)
  (e) with the 6 colors each in their own edge connected region.
  (1 solution)

(2) Pack the 14 pieces left after the second smallest piece is removed.
(76378 solutions)

Kadon Enterprises, Inc. offers a physical version of Hex-Pave in laser cut acrylic here: http://www.gamepuzzles.com/tiling4.htm#HPv.

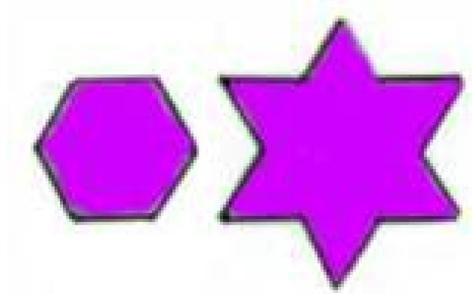


StarHex-14™

The Beauty of Polyform Sets

A presentation in 14-line sonnets by Kate Jones for G4G14

kate@gamepuzzles.com



A product of Kadon Enterprises, Inc.

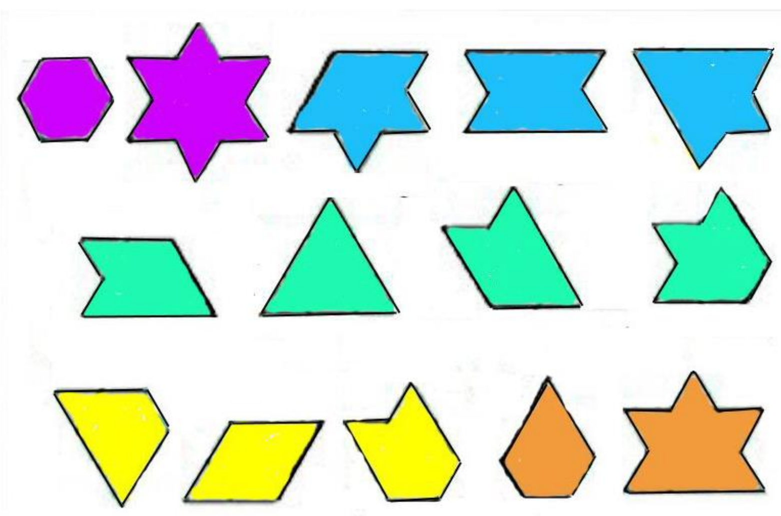
1.

Four decades of design and puzzle art
Have filled my life and overjoyed my heart.

I thank you all for treasuring my creations,
We do have fans in many different nations,

Whence ever more inventors join our line:
We publish their designs, too, not just mine.

From *Theo Geerinck*'s brilliant creative mind
New polyforms—*his polystars*—we find.



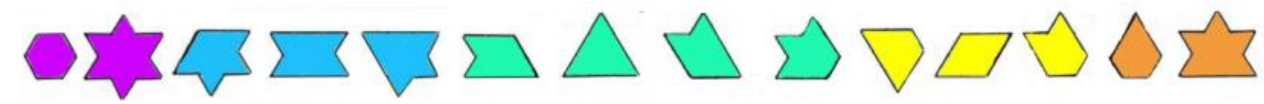
Their stellar highlight, at this grand event,
Is just the latest stage in their ascent.

Dear Martin Gardner's writings were the gate
Through which I found my fortune and my fate.

And by a lovely match of tiles and times
The number fourteen has inspired these rhymes:

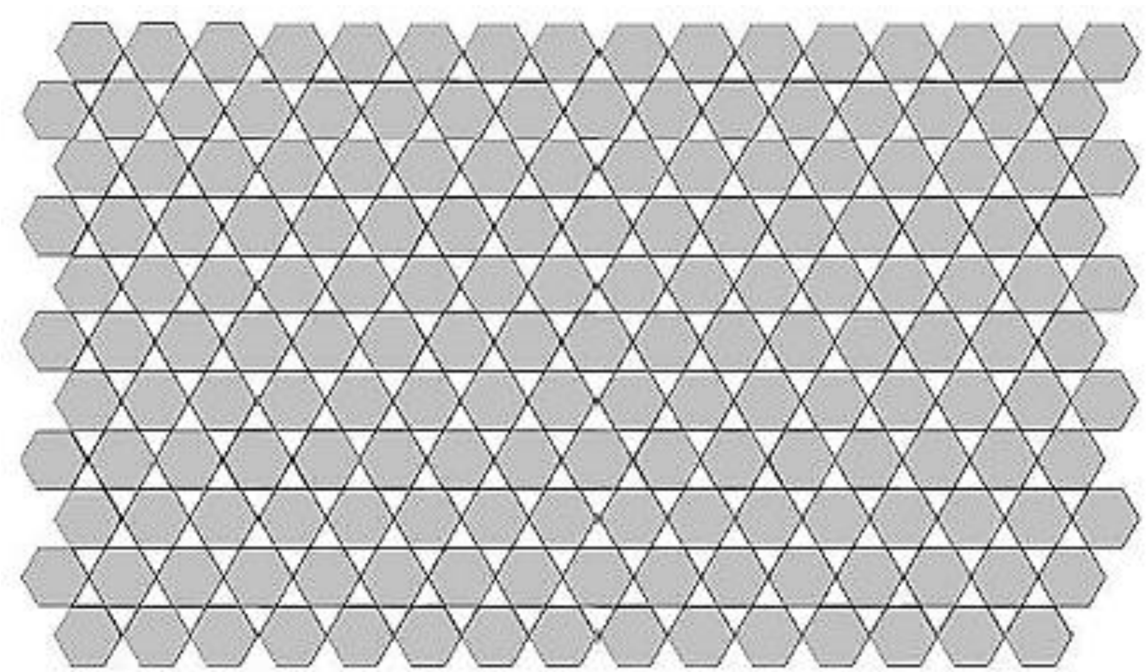
2.

The Cosmos spreads before us points of light,
Each tiny dot a fire immensely far,
A galaxy of giants, dim or bright,
A burst of energy we call a star.



Now let us model constellations fair
With puzzle shapes that mimic starry skies,
And let each star from none to six points share,
No two alike, that fourteen figures rise.

They plot upon a classic field—behold
How triangles and hexagons do spill
An infinite array. A tale is told
How polystars this symmetry will fill.



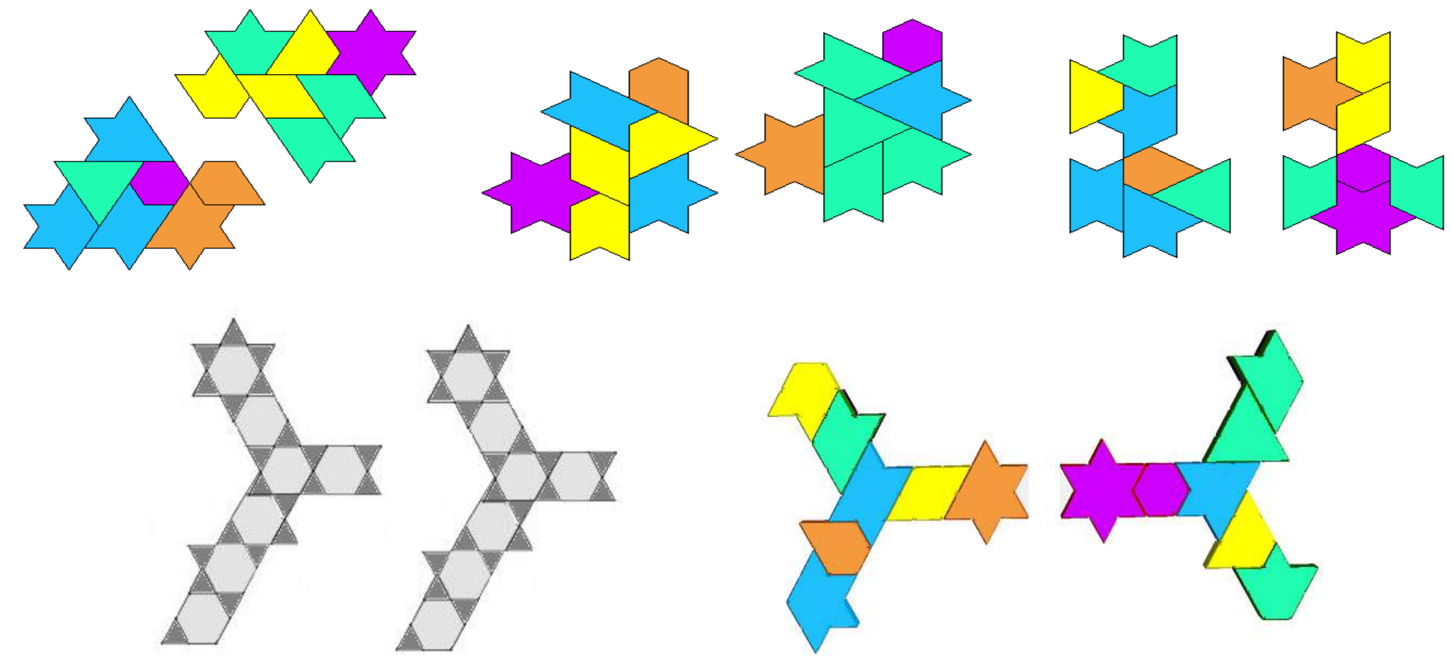
A classic wallpaper pattern

Now may their hues not touch, a solver's dream,

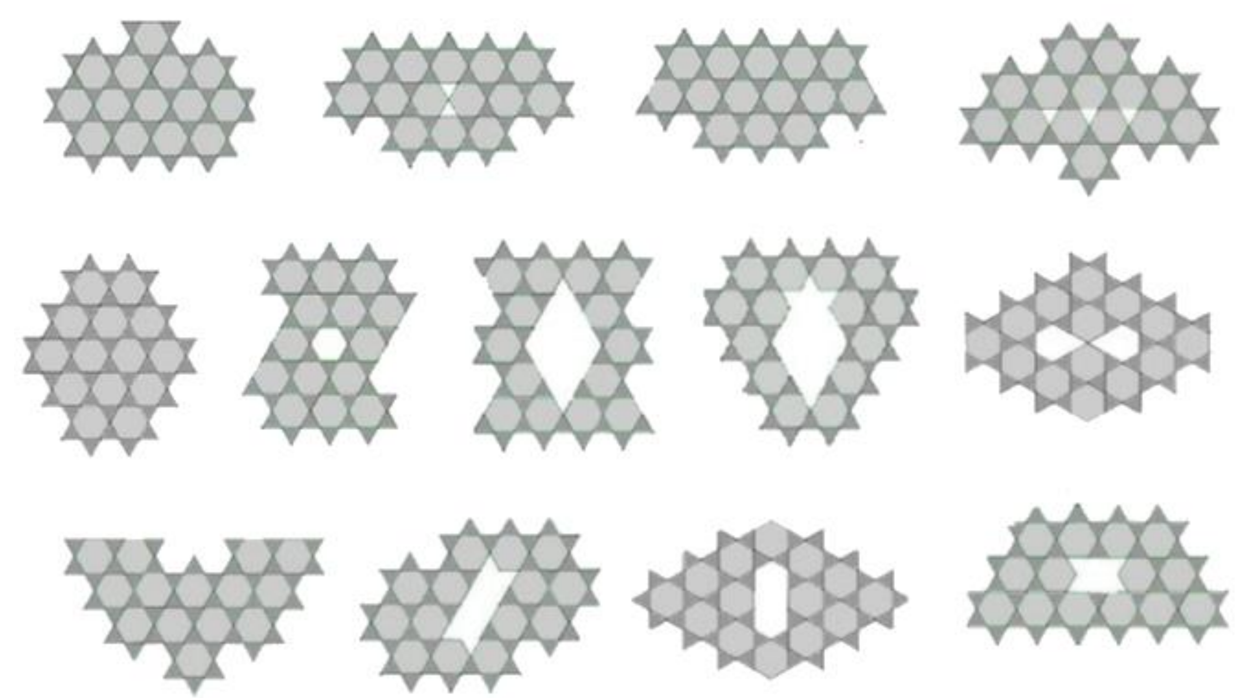
Or group each color as a single beam.

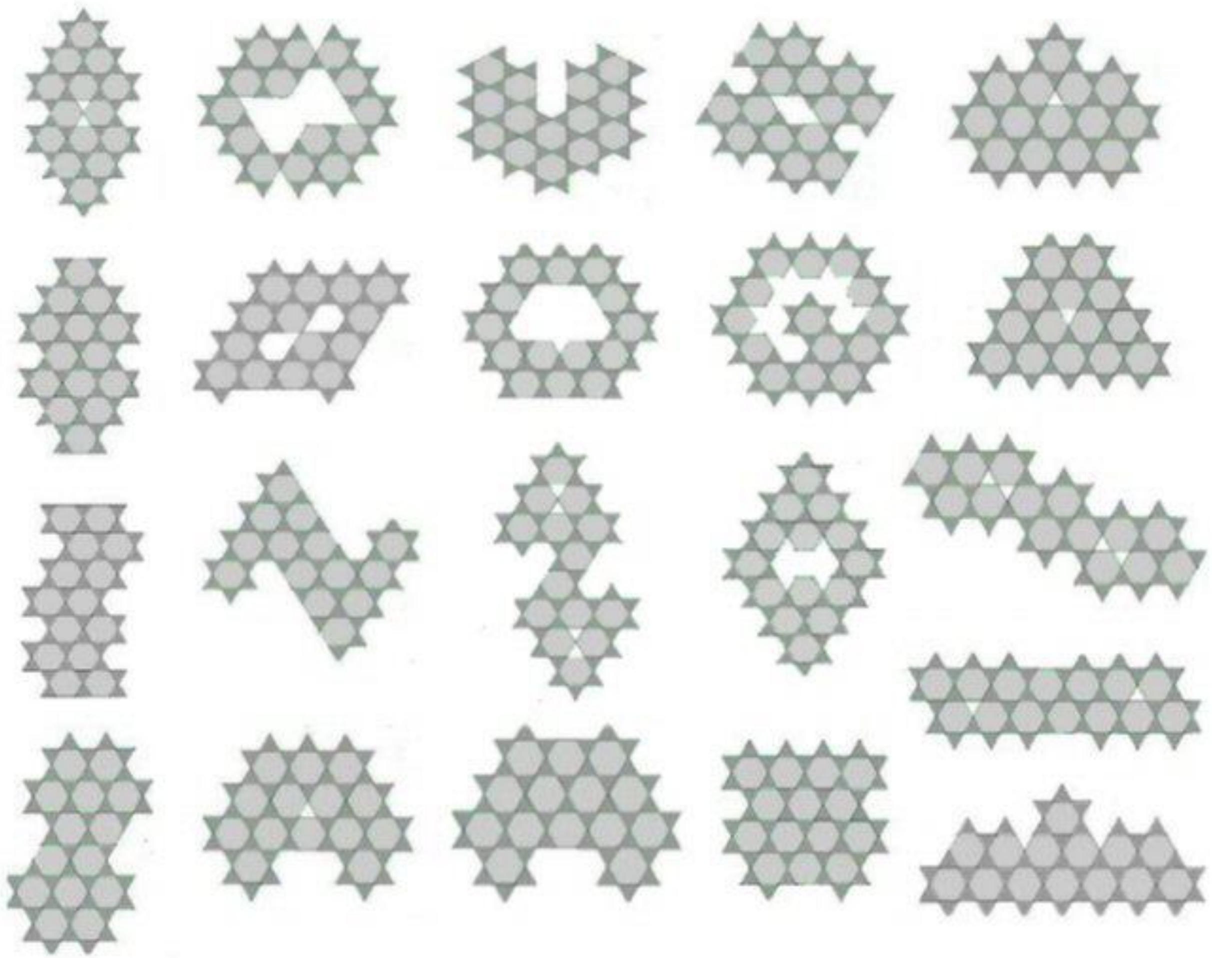
3.

Then let the puzzler build some pretty twins,
Binary pairs or splendid symmetries

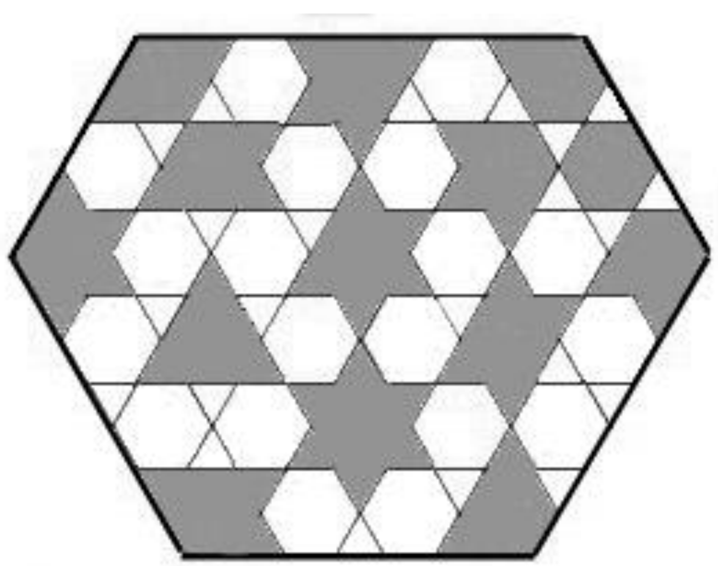


Where every shape a glorious orbit spins
The eye and mind with mirror forms to please.

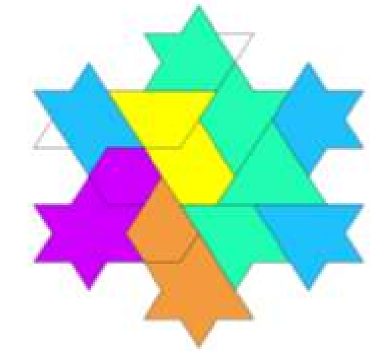




Now capture all your stars within a hive
That only corners touch, a mighty feat,



Or join just thirteen so that stars arrive,
2 points less, yet balance mirrored sweet.



This marvelous set in Kastellorizo earned
The Archimedes prize for puzzle fame.

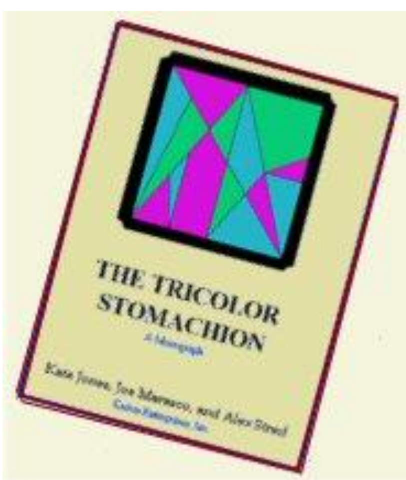
With Archimedes, too, a fourteen turned
Into an awesomely historic game.

We celebrate today the old and new
As polyforms define the good and true.

Tricolor Stomachion

The *other* 14-piece puzzle!

See the Monograph with full analysis of all 1072 solutions, www.gamepuzzles.com/tsm.htm



Three Puzzles on a Familiar Theme

Submitted by Justin Kalef, June 2022

Explanatory note: Some time ago, I noticed an original copy of Accolade's classic video game, 'Hardball!', offered for sale on eBay. The listing brought back fond memories of playing the game on my Commodore 64 almost forty years ago, before enjoying a visit from my grandfather. I made the purchase from the somewhat mysterious seller, and soon received my package with the simple return address 'Haddon Hall, UK'. The game was in good condition (other than the soundtrack, which appeared to have been modified), but I also found three sheets of handwritten notes stuffed into the box. Contacting the seller to return them proved impossible. I therefore took the liberty of reading them, in hopes of tracking down the writer. From what I could tell, all three of the sheets were excised from some longer work of literature, but I can't tell whether it was a novel, a screenplay, or something else. One would imagine that the author – one S. Morgenstern – had some professional writing experience, but I haven't been able to find anything else written by him or, perhaps, her. Still, the three fragments I discovered in the game box do hold some interest for those interested in solving logic puzzles. My own guess, admittedly unsubstantiated, is that S. Morgenstern became worried at some point that including the logic puzzles might alienate some of the intended audience, and therefore cut them. Still, they do seem to be just the sort of thing that might amuse some devotees of the Gathering for Gardner. All three fragments involve twists on the familiar 'knights and knaves' puzzles, and their odd setting within Morgenstern's fantasy narrative may be charming for some. I therefore present these three 'objets trouvés', if you will, as my contribution to the G4G14 gift exchange. I have also worked out what I think are solutions to all three problems, and would be glad to pass those solutions along to any interested parties who write to me.

Best to all,
Justin Kalef

Fragment 1

"Just one moment," said Vizzini, standing up theatrically. As the Man in Black and Buttercup turned in surprise, he started cackling. "Did you really think that I, one of the great geniuses in all history, would be tricked so easily? Did you imagine that you were the only person on Earth with the foresight to build up a tolerance to iocane powder? You fool!"

The Man in Black stepped up to the table again, noticing an unsteadiness in his feet. "Clever. But I'm afraid it's of no consequence anyway. The Princess is safe with me and, as you admitted, you are no match for my physical strength."

This somehow made Vizzini giggle. "Don't you see? While you were distracting yourself with your little game with the powder, you failed to notice the unusual color of the wine. Do you taste that bitter flavor now? The entire bottle has been treated with the extract of phlephm root. It has a rather paralyzing effect, don't you think? You should be feeling a numbness in your knees about now. I suggest we sit."

The Man in Black did find it more difficult to move his legs. As he sat on his log again, Vizzini continued: "Now, as everyone knows, the only perfect antidote to phlephm root extract is the juice of the phlephm berry. I don't suppose you have any phlephm juice with you? I thought not. I, however, took the antidote minutes before you arrived. I foresaw everything! And if *you'd* like to take some before it's too late, I advise that you watch and listen closely." With that, Vizzini drew a gold cup and a silver cup out of his bag. On the gold cup were engraved the words 'One of these two cups is from Sicily'. On the silver one was engraved 'The gold cup is from Sicily.' "These cups are two of my four favorites," he explained. "One is Sicilian, one is from Florin, one is from Spain, and one is from Greenland. Each is made of a different metal."

"I see only two," said the Man in Black, noticing that he could no longer move his thighs.

"I take at most two with me when I travel", said Vizzini. "But I always make sure that any statement engraved on any of them is true or false, according to a scheme. The Spanish one, you see…"

"No need to say more," said the Man in Black, feeling a strange tingling in his lower back. "I've known too many Spaniards."

"I was going to say," replied Vizzini, "that I make sure that any statement engraved on the Spanish one is true, in memory of one of my frustratingly principled recent employees. Now, of course, I make sure that any statement on the Sicilian cup is false, because – as you should know by now! -- one should never go in against a Sicilian when…"

"Please," begged the Man in Black, "I'm already feeling a tightness in my chest. A little less exposition and more haste, I beg you."

"Fine. I always ensure that the statements on the cups from Spain and Greenland are true, and that those on the cups from Sicily and Florin are false. Into the cups bearing false statements, if either or both are here, I will pour wine laced with ospion, a rather unpleasant toxin. One drop, and you'll be lying here in terrible agony for two full days as you die slowly. There is no known antidote. Into the cup from Spain, should it be among these two, I will pour wine containing enough phlephm juice to cure your current paralysis instantly. And into the cup from Greenland, should it be here, I will pour wine containing the only other antidote to phlephm root extract: tea made from phlephm leaves. A sip of that tea, and you will slip instantly into a very pleasant sleep for twenty-four hours, after which you will awaken to find that the paralysis is gone, I am miles away, and the Princess is dead."

As Vizzini turned away with the cups and fiddled with hidden bottles, the Man in Black asked how he could know that both cups would not be poisoned. "You can't!" retorted Vizzini, returning them to the table. "But consider: given what you know of me, am I the sort of person who would boldly give his enemy a chance to recover, and then perhaps to overpower me and force me to drink my own poison? Or am I the sort of man who would place his enemy in an impossible situation? Now, it might be sensible to imagine that I would be unsportsmanlike. But clearly, that cannot…"

"I'd rather rely on logic than psychological speculations," interrupted the Man in Black, feeling a growing coldness in his fingers. "But you seem not to have supplied me with enough information to make that possible."

"On the contrary," cried Vizzini, "I'm giving everything away! Look at the backs of the cups!" And he twisted them around. On the back of the gold cup was engraved 'This cup is from Sicily if, and only if, the silver cup is from Florin.' On the back of the silver cup was engraved 'This cup is from Sicily if, and only if, the gold cup is from Greenland.' The Man in Black was glad that, in addition to the arts of navigation, piracy, mountaineering, and swordsmanship he had now studied intensely for many years, he had undertaken rigorous training in deductive logic. And he was relieved to recall that iocane powder leaves one unable to speak falsely for several hours, even if one is immune to its deadly effects. But he knew that, in less than a minute, he would lose the ability to reach for either cup.

<u>Fragment 2</u>

Westley: *… and I have been Roberts ever since. Except now that we're together, I shall retire and hand the name over to someone else. Is everything clear to you?*

Buttercup: *Not yet. I just don't see how you could trust a man – Ryan, I think you called him – after he'd lied to you so often. I mean, he'd given you a false name, and said he was probably going to kill you every morning. How could you even sleep with that fate hanging over you?*

Westley: *Well, I was quite worried at first, but some members of the crew tried to put my mind at ease right away, telling me that they knew Roberts… er, Ryan, as I know him now… meant me no harm. Others on the crew told me otherwise, but I soon came to understand that those ones were simply lying.*

Buttercup: *The crew members were lying to you?*

Westley: *All day long, I'm afraid. At least, some of them lied on some of the days. But it didn't take that long to sort it all out. You see, each day, they would decide among themselves which would tell the truth and which would lie. One that had been decided, they would keep it up all day, either lying all the time or telling the truth all the time until they had gone to sleep. And the next morning, they would meet up secretly and decide who would lie and who would tell the truth on that day. Some of them switched more or less every day, while others stayed as liars or truth tellers for months at a time. One never knew what it was going to be.*

Buttercup (narrowly avoiding a sudden fire with Roberts' help): *It really sounds dreadfully confusing. How did you ever figure out which were lying for the day and which were telling the truth?*

Westley (continuing to slash away the vines blocking the path): *Fortunately, it didn't take me that long to come up with a few questions I could ask the crew every morning that would tell me whom I could trust and whom I couldn't – and even when I knew my crewmates* were *lying, I was also able to get whatever information I needed from them. In fact, I could instantly learn the truth of whatever I was interested in, since the rest of the crew always answered my questions as well as they could. During those three years – three years and a bit, really – I kept a diary of who told the truth and who lied on each day, thinking the pattern of liars and truth-tellers might repeat itself: I mean, I thought there* must *come a day when each member of the crew would lie or tell the truth just as they already had on some* *single day I had already spent with them. But they managed to avoid repeating any such pattern until there were no more new ones to try. Coincidentally, perhaps, that was on the same day that Ryan took me to his room and told me his great secret.*

Buttercup: *What an extraordinary story. But what* did *you ask them each morning to figure out which ones were lying and which were telling the truth? And these crewmates: did they all… how many did you say there were?*

Westley: *I* didn't *say. But why don't you try to figure that out? I'd hate to spoil the fun for you.*

(Buttercup widens her eyes and nods, then immediately furrows her brow in a confused look. Turning, she falls with a shriek into a lightning sand pit).

Fragment 3

Westley demands the gate key. Yellin at first denies having such a key, but changes his tune when Inigo tells Fezzik to tear Yellin's arms off. "Oh, you mean *this* gate key," he quickly replies. "You may have it, but you won't know which gate to open without me. One leads safely into the main floor of the castle, and the other five lead to the various levels of the Zoo of Death that hides beneath it."

"We don't need you for that," says Inigo. "Now that we have your key, it won't take us long to try all six doors."

Yellin laughs. "No, not long at all, if there's anything left of you by the end. If you try the door that leads to Level Five, for instance, you'll have to contend with Prince Humperdinck's green speckled recluse – a cherished pet spider that makes its home quite near the handle of the door, in back. You'd never know it's there unless you were to disturb it by trying the handle. If you were unwise enough to do so, you'd find out why they say that, compared with the green speckled recluse, the black widow is a rag doll. And these signs you see above the six doors? They're not as helpful as they look, since only the one that leads to the main floor is correct."

"So all but one of the six signs has a lie written on it?"

"That's not such a terrible average when you consider that this system has kept us safe," replies Yellin. "But you see my point. This key may open the door to the castle, but you'll need my help if you want to get in safely, just as I need your help to ensure my safe return to Holland once the Prince discovers my betrayal. I'll need a thousand gold florins, please: Dutch guilders would be my preference, but I'll accept whichever you happen to have between you. Without that many gold coins to jog my memory, I'm afraid I won't be able to recall just which gate you should open."

At the words 'jog my memory,' Inigo sees in a flash – but still too late – what's about to happen. Before he can shout out his warning, Fezzik delivers Yellin a blow that leaves him quite unconscious and mostly dead. There is no time for another trip to Miracle Max, and the three adventurers have no more gold coins, anyway. Instead, they must rely on Westley's logical skills to make their way inside the castle in time to prevent the wedding and rescue Buttercup. The gates are all helpfully marked with letters (A to F from left to right), and the signs above them read as follows:

Sign over Gate A: *This gate and Gate E lead to Levels 4 and 5 of the Zoo of death, in some order.*

Sign over Gate B: *Gate F does not lead to the main floor of the castle.*

Sign over Gate C: *No gate to the left of this one leads to Level 4 of the Zoo of Death*

Sign over Gate D: *This gate leads either to Level 1 or Level 3 of the Zoo of Death.*

Sign over Gate E: *This gate leads to the main floor of the castle.*

Sign over Gate F: *Gate D does not lead to the main floor of the castle or to Level 3 of the Zoo of Death.*

Westley, thanks to his extensive training in logical thinking, figures out a trick for solving the problem quickly. Inigo opens the door he indicates, and they make their way to the main floor just as Prince Humperdinck demands that the Impressive Clergyman rush to the end of the wedding ceremony.

Jumping Julia Mazes

By Daniel Kline

The Julia Robinson Mathematics Festival (jrmf.org) has been using Jumping Julia mazes for over a decade at our math festivals, events designed to share fun, meaningful math with K12 students. Our Jumping Julia mazes are based on Number Mazes, which were first published by Sam Loyd on April 24^{th}, 1998 and popularized by Robert Abbott.[1] At our in-person festivals, Jumping Julia mazes are displayed on large floor mats that students can literally jump through to solve.

Students on a Jumping Julia maze mat at a 2019 math festival

A Jumping Julia maze is a grid of numbers, like the one below. To solve a Jumping Julia maze, you need to follow these rules:

1. Start on the top left square.
2. The number you are on tells you how many squares you must jump.
3. You can only jump in a straight line left, right, up, or down. You cannot move diagonally or in an L-shape.
4. Your goal is to reach the bottom right corner.

3 Start	2	3	2
2	1	2	1
2	2	2	2
1	3	1	Goal

[1] http://cs.gettysburg.edu/~tneller/rjmaze/index.html

Because COVID prevented us from hosting in-person festivals, we wanted to find another way to share Jumping Julia mazes. We created an online app for Jumping Julia mazes that is free for everyone and has 45 puzzles of different sizes and difficulty. You can find our Jumping Julia app here: www.jrmf.org/activities/jumping-julia.

Here's a sample of some of our favorite Jumping Julia mazes:

Puzzle 1

3 Start	2	2	1
3	2	1	3
1	1	1	1
1	3	1	Goal

Puzzle 2

2 Start	1	3	3
2	1	3	1
2	2	2	2
1	3	2	Goal

Puzzle 3

2 Start	3	4	3	1
3	1	4	2	4
3	1	2	1	3
3	1	3	2	3
4	2	1	2	Goal

Puzzle 4

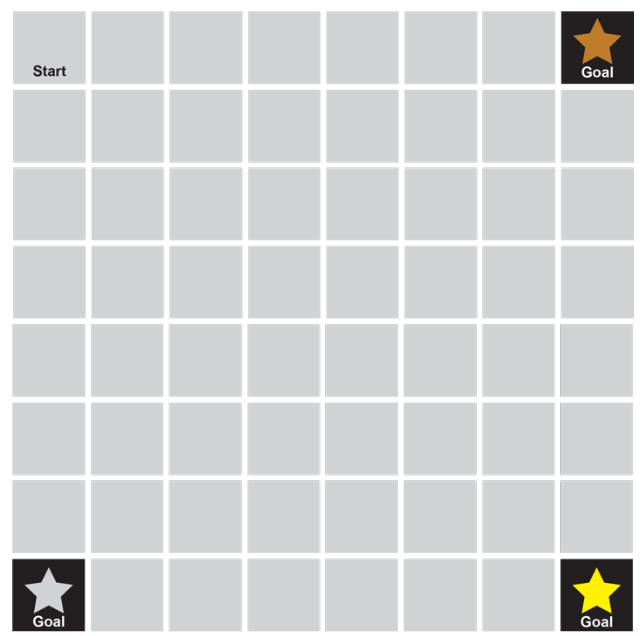
2 Start	5	4	4	3	3
4	1	3	4	1	3
5	2	1	5	1	4
3	4	2	1	1	4
4	5	5	1	2	1
2	2	2	5	4	Goal

As you explore our Jumping Julia mazes, here are some questions to think about:

1. Can you find more than one solution for each maze?
2. What is the fewest number of jumps you need to solve each puzzle?
3. Can you find a solution that visits every square?
4. Are there squares that aren't helpful for solving a puzzle? What makes them unhelpful[2]?
5. Can you find a strategy that helps you solve any Jumping Julia maze quickly and efficiently?

Recently, we've been able to host in-person festivals again, and we're currently trying to find the best way to turn our Jumping Julia mazes into a main attraction. Because of this, we wanted to design a much larger maze with multiple goals, and I wanted to leave the puzzle design challenge we've been working on here for you all to grapple with:

1. Make an 8 x 8 Jumping Julia maze with 3 goals in the non-start corners.
2. Make each goal a different difficulty (you get to decide what "difficulty" means).
3. Include at least one "loop" trap. You can see a loop trap made out of four 2's in the example puzzle on page 1.



Want your own Jumping Julia maze mat? Purchase one through our website, or learn how to make your own for free! Learn more here: www.jrmf.org/maze-mat.

[2] Words like "helpful," "unhelpful," "quickly," and "efficiently" are left intentionally vague. Each of these terms may mean different things to different people, and we leave it up to you to decide which definition is most meaningful to you!

Are You Smarter Than Google?

Chapter 8 of *Can You Outsmart an Economist?* by Steven E. Landsburg

Chapter 8

ARE YOU SMARTER THAN GOOGLE?

In the years I've been blogging at TheBigQuestions.com, I haven't shied away from controversy. Religion, politics, and manners are standard fare, though I try not to post unless I have something at least a bit novel to say. As a result, I'm usually not preaching to any particular choir, which means I risk offending every variety of knee-jerker.

I've gotten used to being called a radical socialist, a mindless liberal, a heartless conservative, and a reactionary mooncalf. But in all my years of blogging, no post has inspired more vitriol than one titled "Are You Smarter Than Google?".

In fact, it's not even close. This post generated many thousands of responses, both on my own blog and others, a great many of them demanding that I be fired, publicly humiliated, and/or banned from the Internet. I don't delete comments, even when they're very strongly worded, unless they're extremely abusive and/or quite thoroughly devoid of intellectual content. In this case, I deleted many hundreds.

What was the content of this post? It was the following brain teaser:

1

ARE YOU SMARTER THAN GOOGLE?

There's a certain country where everybody wants to have a son. Therefore each couple keeps having children until they have a boy; then they stop. What fraction of the population is female?

Well, of course you can't know for sure, because maybe, by some extraordinary coincidence, the last 100,000 couples in a row have gotten boys on the first try, or maybe, by an even more extraordinary coincidence, the last 100,000 couples have had to try eight times before succeeding.

Therefore (as I told my readers in the original blog post), the question is meant to be answered *in expectation*, which means this: If there are a great many countries just like this one, what fraction of the population is female in the average country?

This problem has been around, in many forms, for at least half a century, but it keeps finding new life. I found it in a children's puzzle book when I was about ten years old, and (much more recently) Google has used it to screen job candidates. The official answer — that is, the answer I found in the back of that puzzle book, and the answer Google reportedly expected from its job candidates — is simple, clear, and wrong.

And no, it's not wrong because of small real-world discrepancies between the number of male and female births, or because of anything else that's extraneous to the spirit of the problem. It's just wrong. The correct answer, unlike the expected one, is not so simple.

So: are you smarter than the folks at Google? Before you read ahead, what's your answer?

I'll wait....

Ready now?

Okay, let's continue.

The answer Google seems to have expected is the same answer I gave when I first saw this problem long long ago. It goes like this:

> Each birth has a 50% chance of producing a girl. Nothing the parents do can change that. So each individual child is equally like to be male or female, and therefore, in expectation, half of all the children are girls.

I'll give you another chance to take a break. Before you read ahead, what's wrong with that reasoning?

Ready?

Okay, then:

Actually, most of it is right. Each birth has a 50% chance of producing a girl — check! Nothing the parents do can change that — check! So, each individual child is equally likely to be male or female — check!

But it does not follow — and in fact is not true! — that in expectation, half of all children are girls.

What *does* follow is that, in expectation, the number of boys and the number of girls are equal. But that's not at all the same thing.

To see why not, try this much easier problem:

2

EGGS AND PANCAKES

Every day I flip a coin to decide what to have for breakfast. If the coin comes up heads, I have two eggs and one pancake. If it comes up tails, I have two eggs and three pancakes. On average, what fraction of my breakfast items are pancakes?

THE WRONG SOLUTION: On the average day (in fact each and every day day!) I have exactly two eggs.

On average day, I also have two pancakes (two being the average of one and three). So on average, the number of pancakes is equal to the number of eggs.

Therefore on average, half my breakfast items are pancakes.

Except for the final sentence, all of that is true but most of it is irrelevant. I do in fact have two pancakes on the average day, but *that has nothing to do with the question*. Here's the right answer:

THE RIGHT SOLUTION: Whenever I flip heads, $1/3$ of my breakfast items are pancakes. Whenever I flip tails, $3/5$ of my breakfast items are pancakes. The average of those two numbers is $7/15$. The answer to the question, then, is that on the average day, $7/15$ of my breakfast items are pancakes.

Here's the analogy:

Imagine many breakfasts	Imagine many countries
At the average breakfast, the number of pancakes is equal to the number of eggs (TRUE!)	In the average country, the number of girls is equal to the number of boys (TRUE!)
Therefore at the average breakfast, the fraction of items that are pancakes is 1/2 (FALSE!)	Therefore in the average country, the fraction of children that are girls is 1/2 (FALSE!)

MORAL: **Two things (be they eggs and pancakes or boys and girls) can be equal in expectation,**[1] **but that tells you nothing about their expected *ratio*.**

The gist of that moral is that the official answer to the Google problem is wrong. But we still have to figure out what's right.

It turns out that the correct answer depends on the size of the country. This is easiest to think about when the country is so tiny that it has just one family. Let's solve that case first; then we'll move on to bigger countries.[2]

Here are some possible configurations for that one family:

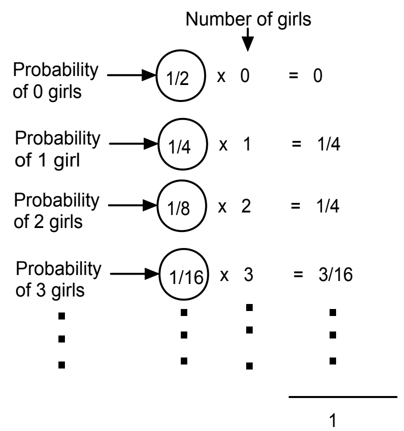
PROBABILITY	CONFIGURATION	FRACTION FEMALE
1/2	B	0
1/4	GB	1/2
1/8	GGB	2/3
1/16	GGGB	3/4

From this, we can see that the number of boys is always exactly 1.

The number of girls, of course, can be anything at all, but we want to know what it is on average. For that, we take each possible number, multiply it by the corresponding probability, and add up, as follows:

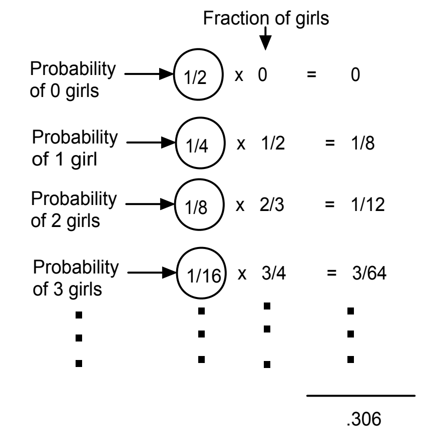
[1] Remember that "in expectation" means the same thing as "on average".

[2] If you — like many of my blog readers — are prepared to object that the one-family assumption is contrary to the spirit of the problem, let me assure you that I agree with you. I'm solving this case first not because it's the most important case, but because it's the easiest. I hope that you, unlike some of my more impatient blog readers, will bear with me.



The numbers in the infinitely long column on the right add up to 1. (If you don't believe me, try adding several terms and you'll see them approaching closer and closer to 1.) That is, the expected number of girls is equal to the expected number of boys — just as we knew it must be.

But to get the expected *fraction* of girls, we need to do the same calculation with fractions of girls instead of numbers of girls. And it comes out like this:



The numbers in the right hand column add up to just about .306, or 30.6%[3]

Now again — that calculation is only correct for a country with just one family. For a country with two families, a similar but more complicated calculation gives an expected fraction of about 38.63%. If you want to see that calculation, you can look in the appendix to this book.[4]

For a country with 10 families, the expected fraction is about

[3] Where did the 30.6% come from? It takes a bit of work. If you remember your calculus, you might be able to show that the sum of the infinite series is actually $log(2) - 1$, which is just about 30.6%. If you don't remember your calculus, I hope you'll take my word for this.

[4] The calculation in the appendix will appeal to the sort of readers who find that thing appealing and not to others. Fortunately, the main point here is not to follow the complicated calculation, but to understand why it's necessary. That is, the important thing is not so much to understand the right answer as to understand why the "obvious" answer is wrong.

47.51%. For a country with 100 families, it's about 49.75%. For a country with 1000 families, it's about 49.98%. For a country with 5000 families, it's about 49.995%. For a country comparable to the United States, with about 100,000,000 families, the expected fraction is about 49.999999975%.

You might be tempted to say, "Aha! Surely there's no important difference between 49.995% and 50%. So the official reasoning is correct after all!"

Hold on there! First of all, even if the correct answer were *exactly* 50%, the official reasoning would still be entirely wrong. We don't generally give full credit (or even partial credit) for bad reasoning that just happens to get the right answer.

Besides, who says there's no important difference between 49.995% and 50%? Try telling that to Al Gore, who got 49.995% of the Bush/Gore vote in Florida in the year 2000, and thereby lost the presidency of the United States.

Or if that doesn't convince you, try this variation, where the official reasoning will lead you neither slightly astray nor moderately astray or even hugely astray, but *infinitely* astray:

3

THE GOOGLE PROBLEM REDUX

There's a certain country where everybody wants to have a son. Therefore each couple keeps having children until they have a boy; then they stop. What is the ratio of boys to girls?

This differs from the original Google problem by asking about the ratio of boys to girls, rather than the fraction of girls in the population.

Again, the answer in any one country could of course be just about anything, so we need to specify that the question is to be answered *in expectation*, or in simpler words *on average* over many such countries.

SOLUTION: There's always some chance — perhaps a tiny chance, but still some chance — that every single family has a boy on the first try. If that happens, there are no girls, so the ratio of boys to girls is infinite.

To get the *expected* ratio, we have to average over all possibile ratios, including infinity. That average is infinity.

If you said that the answer was $1/2$, you were infinitely wrong.

It turns out that a lot of people — and especially, I suppose, the sort of people who like to solve brain teasers on the Internet — have seen some version of this problem before, and have had the (correct) insight that in expectation the number of boys and the number of girls must be equal. Some of them tend to feel pretty proud of that insight, which makes them exceptionally reluctant to admit that it fails to solve the problem.

I'd intended to blog twice on the subject — once to pose the puzzle and once to reveal the answer. Instead, the discussion ended up stretching over six blog posts. You can find links to all of them at www.TheBigQuestions.com/google.html.

A lot of readers fell into the trap. A lot of those defended their answers vigorously, then gradually saw the light as I and other commenters pointed out their errors. Those people learned something, and many of them were delighted. That delighted me, too.

Others brought up interesting and valid new twists. Here are a few examples:

- My analysis assumes that all families have finished reproducing. What if we take a snapshot *before* the last family gets its son? (Answer: It depends on when you take the snapshot. But in no case is the expected fraction of girls equal to 1/2.)
- What if you count the parents and not just the kids? (The answer changes, but it's still not 1/2.)
- What if the country's population is literally infinite? (Answer: Then there are infinitely many girls and infinitely many boys, giving a fraction of infinity over infinity, which is not a number at all, and certainly not 1/2. Besides, who ever heard of a country with an infinite population?)[5]

But others kept returning to the comments section to defend the wrong answer, while a great many others jumped into the fray to help point out their errors — help that was not always appreciated.

The whole thing might have died down in a few days had it not caught the attention of an Internet phenomenon named Lubos Motl. It's been said of Lubos that he's hard to ignore, but it's always worth the effort. I eventually took this advice to heart, but not before we had several rounds of increasingly bizarre correspondence.

Lubos is a physicist by training and a crank by choice. He appears to haunt the Internet twenty-four hours a day from his home in the Czech Republic. When he blogs about physics, he's often clear, accurate and generous with his explanations. The rest of the time he burnishes his reputation as a nut.

That's what he was doing when he announced on his blog that

[5] In a delightfully ironic twist, many of the readers who insisted on assuming the population was literally infinite were the same readers who excoriated me for working through the case of a single-family country, even for illustration, on the grounds that a single-family country is "unrealistic".

the only acceptable answer to the Google problem is 50%, and that you (or in this case I) would have to be a complete idiot to believe otherwise. He gave absolutely no argument to support this position, and repeatedly asserted that no argument was necessary. Those who know him will recognize this as classic crank-mode Lubos.

Because Lubos was quite insusceptible to reason (completely ignoring, for example, a series of simple numerical examples that proved him wrong, and refusing ever to state the secret additional assumptions that he claimed would support his 50% answer), I went a different route and publicly offered to bet him up to $15,000 (and anyone else up to $5000) that a computer simulation (for a country with four families reproducing for 30 generations) — with disputes over interpretation to be settled by a panel of randomly chosen statistics professors from top departments — would prove me right.

At first a dozen readers stepped up to get in on this bet, but they all soon either changed their minds or mysteriously stopped responding to emails.

Then I screwed up.

A reader named Larry suggested a slightly different bet, which I accepted without carefully reading his terms. This bet turned out

to be stacked against me.

I knew that in a country with four families, the expected fraction of girls is about 44%. I therefore agreed to Larry's bet that a series of simulations would show it to be less than 46.5%, leaving a little room for statistical anomalies. But I overlooked Larry's stipulation that we include the parents in the count. This turns out to drive the expected ratio up over 46.5% (though it's still less than 50%).

Having rashly accepted Larry's challenge, I was legitimately on the hook for a $5000 bet I was almost sure to lose. I'd have paid up if necessary, but Larry most graciously suggested that he'd settle for some autographed books.

Hundreds of others refused to take the bet but continued to defend the wrong answer. A happy exception was a reader known to me only as Tom, who started out as a serial repeater of false and tired pro-50% arguments. I (and others) tried patiently pointing out his errors, but he seemed hell-bent on ignoring everything we said — to the point where I eventually lost my patience and said "I'm sorry, but it appears that you are too stupid to think about this brain teaser". To his great credit, Tom responded not by digging his heels in further, but by taking a little time to think — and then returning a few days later with a beautifully reasoned essay that not only explained the right answer but offered a whole new (and completely correct) explanation of why the answer cannot possibly be 50%. He graciously allowed me to share his essay with my readers as a guest poster, and I know from my email that it helped a lot of people see the light.

That happy experience aside, I remain astonished that so many became so emotionally invested in defending the wrong answer to a simple brain teaaser. Clearly, the right answer comes as a surprise to many people. It came as a surprise to me at first! But I still don't quite get why so many people are so resistant to being surprised. Or more to the point: How does someone get so emotionally invested in a simple brain teaser that he is willing to make the same false arguments over and over and over and over and over and over and over again, but not care enough to read and digest the right answer? Perhaps that would be a good puzzle for a book called *Can You Outsmart a Psychologist?*

*   *   *   *

Over many years of teaching, one thing I've learned is that when students don't see the point of pure theory, you can usually snag their attention with an application to sports. (Interestingly, this works best with students who are inclined to dismiss pure theory as "just a game".)

Let us, then, turn to the age-old issue of "hot hands" in basketball. The question is whether basketball players experience good and bad streaks beyond what you'd expect from pure chance. Of course we've got a lot of data on this, but historically a great many people have misinterpreted those data—*precisely* because they didn't understand the issues in the great Google problem controversy.

I'll tell you that story in a moment. But first, let me show you how to make some money.

We'll play a game: One of us flips a coin four times in a row to get three pairs of consecutive H's and T's. For example, if you flip $HHTH$, your three pairs are HH, HT, and TH. If you flip $THTT$, your pairs are TH, HT, TT.

Now: I'll give you a dollar for each HH, and you give me a dollar for each HT.

This is a perfectly fair game, because HH and HT are equally likely. If you doubt me, try writing down all sixteen possible outcomes, and count all the HH's and all the HT's. There are exactly twelve of each:

	Number of HH	Number of HT
$HHHH$	3	0
$HHHT$	2	1
$HHTH$	1	1
$HHTT$	1	1
$HTHH$	1	1
$HTHT$	0	2
$HTTH$	0	1
$HTTT$	0	1
$THHH$	2	0
$THHT$	1	1
$THTH$	0	1
$THTT$	0	1
$TTHH$	1	0
$TTHT$	0	1
$TTTH$	0	0
$TTTT$	0	0
	$TOTAL : 12$	$TOTAL : 12$

If you play this game against an experienced gambler, he or she will quickly realize that it's fair. First, experienced gamblers have a very good sense of facts like "HH and HT are equally likely". Second, if you play long enough, you'll probably both come pretty close to breaking even on average, which tells you that the game is probably fair.

Now try a variation:

4

WANNA PLAY?

Once again, we'll flip four times to get three sequences. We'll count the HH's and the HT's. I'll give you a number of dollars equal to the *percentage* of those sequences that are HH, and you give me a number of dollars equal to the percentage that are HT.

Does that game strike you as fair?

SOLUTION: If you think like so many of my blog commenters, you'll say "Well, HH and HT are equally likely, so on average half of all the HH and HT flips will be HH and the other half will be HT. This is another fair game."

If you do think that way, please contact me. I'd like to play this game against you. Because here are the relevant percentages:

	Number of HH	Number of HT	Percentage of HH
$HHHH$	3	0	100
$HHHT$	2	1	67
$HHTH$	1	1	50
$HHTT$	1	1	50
$HTHH$	1	1	50
$HTHT$	0	2	0
$HTTH$	0	1	0
$HTTT$	0	1	0
$THHH$	2	0	100
$THHT$	1	1	50
$THTH$	0	1	0
$THTT$	0	1	0
$TTHH$	1	0	100
$TTHT$	0	1	0
$TTTH$	0	0	–
$TTTT$	0	0	–

$$TOTAL : 12 \quad TOTAL : 12 \quad AVERAGE : 40.5$$

In each row, the percentage shown is the percentage of all pairs starting with H that are HH. (In the last two rows, there are no pairs starting with HH, so the ratio can't be computed. If our flips produce either of those patterns, no money changes hands.)

The average of all the percentages is 40.5%, which is definitely not at all the something as 50%. On the average play of this game„ I will pay you $40.50 and you will pay me $\$59.50$. In the long run, I make an average of $19 each time we play.

The moral? Two things (in this case HH and HT can be equal in expectation — that is, they occur equally often — but that tells you nothing about their expected *ratio*. Perhaps that moral rings a bell by now.

Now back to the hot hands.

Take a ball player. Put him at a distance from the basket where he makes just about half his free throws. (This distance will be different for different players.)

Have him take four shots. Write down an H for each success and a T for each miss. Repeat with many different players.

If there are no hot hands, then HH and HT should occur about equally often. (That is, after a successful shot, a second success should be no more likely than a miss.) So a good test of the hot hand theory would be to count the HH's and the HT's for all the players, and see whether the totals are roughly equal.

In 1985, a group of researchers (let's call them GVT, because those were their initials) set out to analyze exactly this experiment. Unfortunately, they kept track of the wrong statistic. Instead of asking whether, on average, there are equal numbers of HH and HT, they figured they might as well ask whether, on average, there are equal *fractions* of HH and HT. After all, equal numbers should be the same thing as equal fractions, right? At least that's what so many of my blog commenters thought — and GVT made exactly the same mistake.

So they counted pairs and computed fractions. And, coincidentally, they discovered that among all pairs that start with H, on average just about half were HH and half were HT.

Here's what they figured: The percentages for HH and HT are about fifty-fifty. That's just what you'd expect from a series of coin flips. So foul shots are like coin flips. There are no hot hands.

Here's what they should have figured: The percentages for HH and HT are about fifty-fifty. If these were coin flips, we'd expect them to be about 40.5 and 59.5. We're getting a *much* bigger fraction of HH's on the basketball court than we'd get from a coin flip. The hot hand must be real.

It took almost twenty years before another group of researchers noticed this mistake. Meanwhile, GVT had fooled not only themselves, but a substantial fraction of the economics profession, into believing that their study had rejected the hot hand hypothesis, when in fact it had confirmed it.[6]

[6] This doesn't necessarily mean that the hot hand hypothesis is *true* — it means only that this particular bit of evidence points in that direction. There is other important evidence in both directions, some of it collected (and correctly interpreted) by the same GVT team.

Once again, the obvious can be the enemy of the true. It's "obvious" that if boys and girls are equally likely to be born, then on average, the fraction of girls should be 1/2. It's equally obvious that if you're equally likely to flip HH and HT, then on average, the fraction of HH's should be 1/2. Neither of those things is true. If you insist on believing them, you're an easy mark for coinflipping con men.

Topological Dance Puzzles
Karl Schaffer

Solutions to most problems are found in my paper "Dancing Topologically" in the Bridges Math and Art Conference Archives, 2021. Bridges 2021 Proceedings. Pp 79-86. http://archive.bridgesmathart.org/2021/bridges2021-79.html. Some problems ask you to create a dance phrase to match a knot or link; that can be as much or more fun than the mathematical part of the puzzle!

1. Imagine that a dancer traversing a path on the stage trails behind a string, Fig 1. The dancer continues to move and trail the string until reaching the dancer's starting point P. The dancer is careful never to move directly over a crossing point of previously trailed strings such as point C by moving slightly to one or the other side of the crossing instead, as indicated by one of the solid arrows. In the figure the dancer then proceeds to starting point P by following one of the dotted arrows and connects the two ends of the string to create a loop. Is it possible that a loop created in this way by a dancer dragging a string might in some cases be a knot (that is NOT the unknot)!? Notice that in the figure the fact that earlier strands of the string pass under later strands is indicated by the "break" in the underlying string segment. It turns out that the decision as to which bold arrow the path takes makes no difference as to whether the loop created in Fig. 1 is knotted, or which knot it might form.

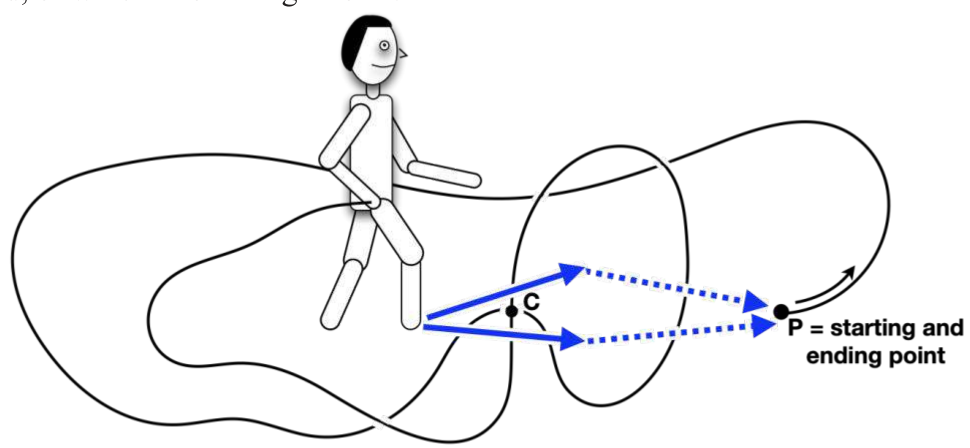


Figure 1

2. Imagine now that each of a pair of dancers trails a string behind themself as they move through a complex floor pattern, never move over a point where two strings already cross, and finally come to rest at either their own or their partner's starting point. Then the string paths will combine to take the form of a knot or link. We say that the diagram produced represents a knot or link that is *duet-* or *2-danceable*. We might turn the exercise around and ask, "Which knot or 2-link diagrams are so danceable by two dancers?"

   (A) Explain why every two-crossing knot is solo-danceable - are they all the unknot? Show that every two crossing 2-component link is 2-danceable.

   (B) Find a three-crossing unknot that is not solo-danceable but is 2-danceable. Perform it with a partner. You should have found in problem 1 that solo-danceable knots are always the unknot, but this problem asks you to find a diagram of an unknot that is not solo-danceable. Generally, knot theorists look for properties of knots that hold for all diagrams of a particular knot, but this shows that danceability is NOT one of those properties!

   (C) Work in a group of four. Find a highly symmetrical way to dance Solomon's knot, Fig. 2, so that two dancers circle clockwise on one link and two circle counterclockwise on the other link. This pattern is called the "Hey for four" in contra dance.

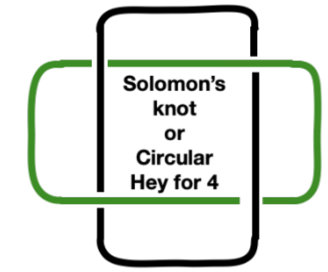


Figure 2

   (D) Work in a group of three or four. Find a 3-danceable pattern for the 3 by 3 Celtic knot (Appendix A) and develop, rehearse, and perform it. It is sometimes helpful to have one person stand out and help direct, which is why a group of four might be helpful.

3. Examine the two diagrams in Fig. 3, which represent the 8-crossing knot that is given the standard label 8_{18} in most knot catalogues. 8_{18} is said to be an 8-crossing knot because 8 is the minimal number of crossings in any of its planar diagrams; however, an *n*-crossing knot may have *n*-crossing diagrams that look very different! Every knot has a *braid* diagram like that for 8_{18} shown in Fig. 4. We imagine that the two strands labeled P are connected, as are the two labeled Q and the two labeled R (without adding additional crossings).

   (A) Explain how 8_{18}'s braid diagram is generated from its standard diagram on the left (hint: pay attention to the three small line segments crossing the knot.

   (B) Find the value *m* such that the 8_{18} diagram is minimally *m*-danceable: every braid with *m* strands is *m*-danceable (can you see why?), but is the 8_{18} diagram also 2-danceable? Explain your reasoning. Remember to pay attention to the direction in which the dancers travel.

   (C) The braid form of 8_{18} also represents the "three-person weave" pattern used in basketball. Watch the video clip https://www.youtube.com/watch?v=DEyULvNXBmo at "Warriors Weave," 2015, in which the Golden State Warriors NBA basketball team uses the three-person weave and explain its relationship to the braid diagram for 8_{18}.

   (D) In a group of three make up a dance phrase that uses the three-person weave.

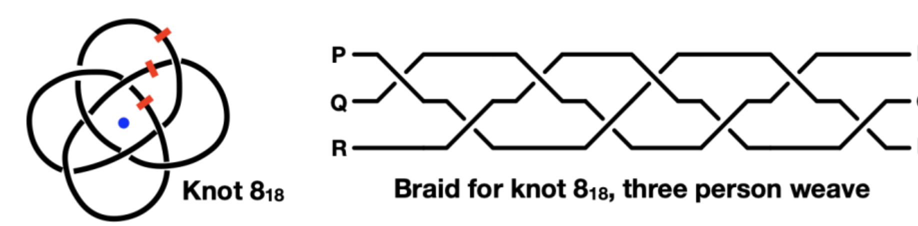
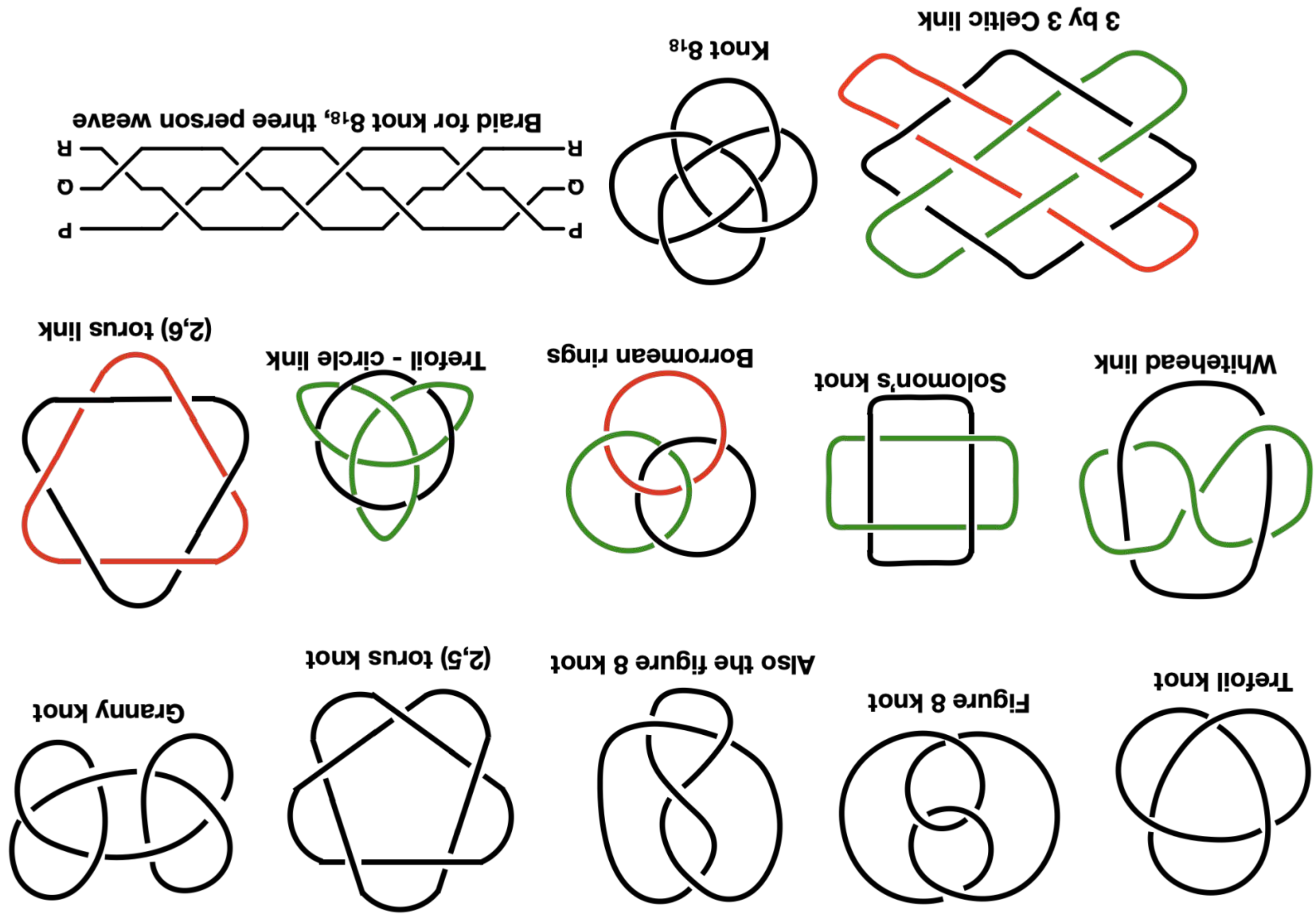


Figure 3: Knot 8_{18} and its braid form, similar to the three-person weave.

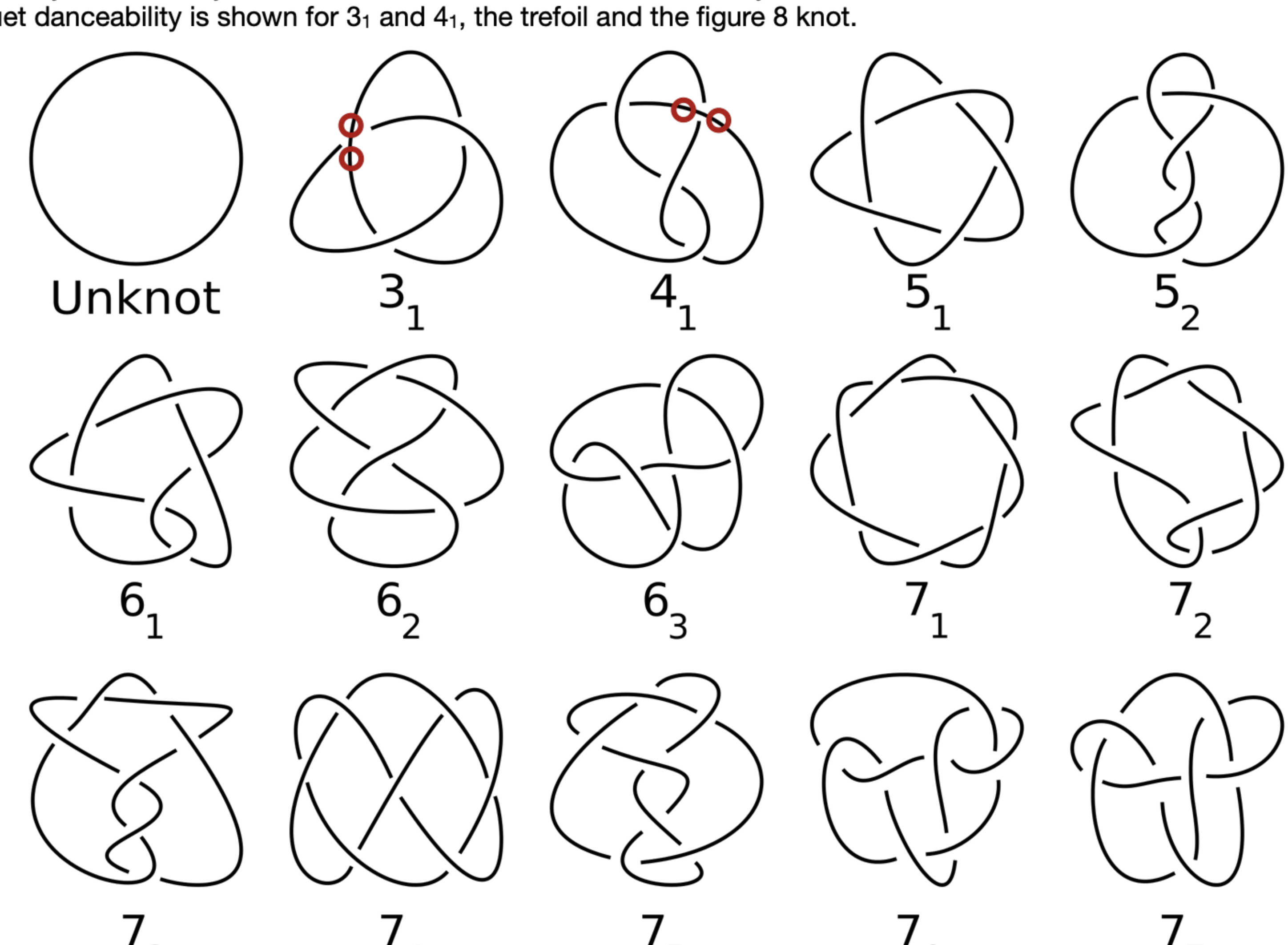
4. For each knot or link drawing in Figures 4 and 5 that you have not yet considered find the minimal *n* such that the drawing is *n*-danceable in each direction. Choose one or two as the basis for a dance phrase. Figure 5 shows diagrams for all prime knots with minimal crossing number less than seven; find "string duet-danceability" as defined in the caption for each.

Figure 4. Knot and Link Examples



Use a pair of red circles OO to show "string duet-danceability," if possible. A knot diagram is string duet-danceable if two dancers can start at a crossing point and move away from that point in opposite directions along the same strand, obeying under and overcrossing rules, until they traverse the entire diagram and meet again. Find which knots are so danceable. Also for each knot not yet considered and each direction on its diagram find *n* and starting points such that the knot diagram is *n*-danceable. Some diagrams may be minimally *m*-danceable in one direction and minimally *n*-danceable in the other direction with *m* and *n* unequal. String duet danceability is shown for 3_1 and 4_1, the trefoil and the figure 8 knot.

Figure 5

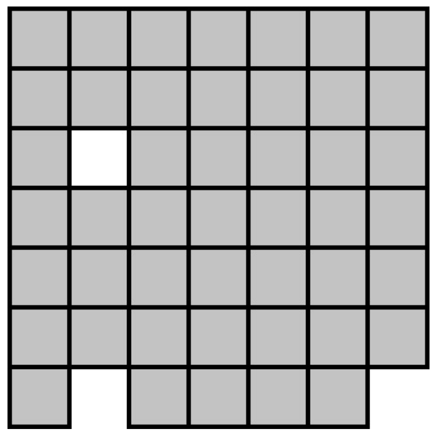


Alexander and Briggs knot notation

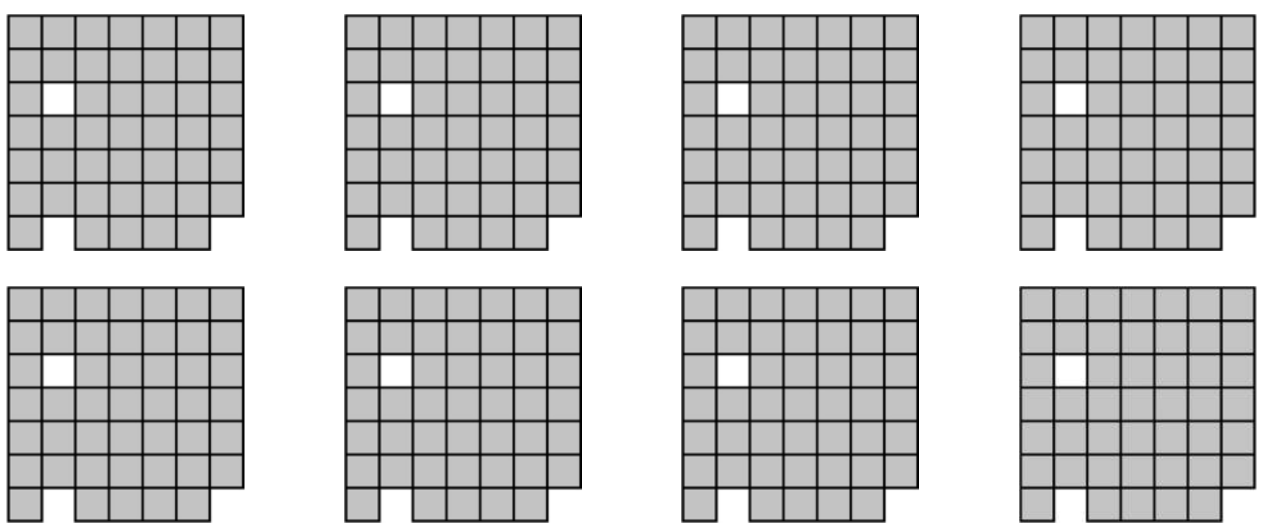
Walk a Crooked Path

by ***Jaap Scherphuis***
G4G14 exchange gift, March 2020

Below is a 7×7 board with three squares removed.



You walk a path from square to square (adjacent horizontally or vertically), without visiting any square more than once. You may start on any square and end on any square, and do not need to visit every square. Find a path that has as many quarter turns as possible, i.e. **what is the most crooked path you could walk?**



This puzzle and its solution can be found on my web site:

https://www.jaapsch.net/g4g/g4g14.htm

Leslie E. Shader's Christmas Card

Les Shader created this Christmas card puzzle around 1986. It was originally submitted and accepted by GAMES magazine before the magazine stopped production.

Permission is granted to reprint this card for personal use, provided that you leave his name on the card.

If you don't want to use MERRY CHRISTMAS, you will have the same solution with HAPPY TRIANGLES.

Leslie E. Shader was a G4G1 through G4G11 attendee. He was a professor of mathematics at the University of Wyoming.

Christmas 1980, Les gave his daughter, Soni, a copy of Martin Gardner's AHA! INSIGHT book.

Soni Shader Huffman introduced this puzzle on the last day of classes before Christmas break to her high school students and taught the counting method found in Martin Gardner's AHA! INSIGHT, pages 9-11. Students in Geometry, Algebra 2, and advanced courses were able to succeed with this puzzle. She was a G4G 11,12,and 14 attendee. She was formerly a high school math teacher, director of the math tutoring center at Northwestern College of Iowa, and a homeschool parent. She now works for AptarePrep.com. Her current interests are antique advertising puzzles and rattleback spinning toys.

Timothy Huffman is a math and actuarial science professor at Northwestern College of Iowa and owns AptarePrep.com. He was a G4G 12 and 14 attendee. He currently collects antique computational devices.

A Christmas Card by Leslie E. Shader, circa 1986.

How do I wish thee a Merry Christmas?

Let thee count the ways.

Count the different paths to say "MERRY CHRISTMAS!" The path may travel UP, DOWN, LEFT, RIGHT, or DIAGONAL.

M
MEM
MEREM
MERRREM
MERRYRREM
MERRY YRREM
MERRY C YRREM
MERRY CHC YRREM
MERRY CHRHC YRREM
MERRY CHRIRHC YRREM
MERRY CHRISIRHC YRREM
MERRY CHRISTSIRHC YRREM
MERRY CHRISTMTSIRHC YRREM
MERRY CHRISTMAMTSIRHC YRREM
MERRY CHRISTMASAMTSIRHC YRREM
!

A Christmas Card by Leslie E. Shader, circa 1986.

Leslie Shader's Christmas Card Solution

This solution uses the counting method from Martin Gardner's **AHA! INSIGHT**, pages 9-11. While a combinatorical solution may be more elegant, I have chosen to use this method because it doesn't require advanced skills and is accessible to high school students.

The solution matrix has been abbreviated for space, but it is obviously symmetric.

1. Start by numbering the M's with a 1.
2. Proceed with the method from **AHA!** counting the number of ways to get to E and R.
3. The difficulty of this puzzle increases because of the double R's. A letter R may appear as the first R or the second R. The solution has two numbers listed for the R's. The first number is the number of ways to arrive at that R as the first R in MERRY. The second number is the number of ways to arrive at that R as the second R in MERRY.
4. 3 R's have a second number of 0 because they are not adjacent to a Y.
5. Proceed with the **AHA!** method; there are no further complications without more double letters.

Beginning of the solution matrix using step 1

		M 1		
	M 1	E	M 1	
M 1	E		E	M 1

Beginning of the solution matrix using step 2. Getting to E

		M 1		
	M 1	E 1+1+1=3	M 1	
M 1	E 1+1=2		E 1+1=2	M 1

A Christmas Card by Leslie E. Shader, circa 1986.

Beginning of the solution matrix using step 2. Getting to R, as the first R.

			M 1			
		M 1	E 3	M 1		
	M 1	E 2	R 2+3+2=7	E 2	M 1	
M 1	E 2	R 2+2=4		R 2+2=4	E 2	M 1

Beginning of the solution matrix using step 3. Getting to R, as the second R.

				M 1				
			M 1	E 3	M 1			
		M 1	E 2	R 7 0	E 2	M 1		
	M 1	E 2	R 4 7+4+4+2=17	R 4 2+4+7+4+2=19	R 4 7+4+4+2=17	E 2	M 1	
M 1	E 2	R 4 4+2+4+2=12	R 2 4+4+4+2=14		R 2 4+4+4+2=14	R 4 4+2+4+2=12	E 2	M 1

=

A Christmas Card by Leslie E. Shader, circa 1986.

Solution matrix, page 1

								M 1
							M 1	E 2
						M 1	E 2	R 4 12
					M 1	E 2	R 4 12	R 2 12
				M 1	E 2	R 4 12	R 2 12	Y 36
			M 1	E 2	R 4 12	R 2 12	Y 36	72
		M 1	E 2	R 4 12	R 2 12	Y 36	72	C 144
	M 1	E 2	R 4 12	R 2 12	Y 36	72	C 144	H 288
M 1	E 2	R 4 0	R 2 10	Y 34	70	C 142	H 286	R 574

A Christmas Card by Leslie E. Shader, circa 1986.

Solution Matrix, page 2

					M 1	
				M 1	E 3	M 1
			M 1	E 2	R 7 0	E 2
		M 1	E 2	R 4 17	R 4 19	R 4 17
	M 1	E 2	R 4 12	R 2 14	Y 81	R 2 14
M 1	E 2	R 4 12	R 2 12	Y 38	157	Y 38
E 2	R 4 12	R 2 12	Y 36	74	C 305	74
R 4 12	R 2 12	Y 36	72	C 146	H 597	C 146
R 2 12	Y 36	72	C 144	H 290	R 1177	H 290
Y 36	72	C 144	H 288	R 578	I 2333	R 578
72	C 144	H 288	R 576	I 1154	S 4641	I 1154
C 144	H 288	R 576	I 1152	S 2306	T 9253	S 2306
H 288	R 576	I 1152	S 2304	T 4610	M 18473	T 4610
R 576	I 1152	S 2304	T 4608	M 9218	A 36909	M 9218
I 1150	S 2302	T 4606	M 9214	A 18432	S 73773	A 18432

A Christmas Card by Leslie E. Shader, circa 1986.

Martin Gardner would sometimes wrap puzzles inside stories he concocted, such as with the book "The Numerology of Dr. Matrix." The following puts my favorite puzzle in that tradition:

The Accountant

by Barney Sperlin

Richard rapped on the Captain's door and heard a faint welcome from inside. He swung it open to find his boss looking up from behind stacks of papers and file folders. The small office reminded him of a university faculty suite, rather than a police precinct.

Richard shut the door quickly and strode up to the front of the desk. "I think we're in, sir!"

Captain Marlow frowned from behind his half-rim glasses. He glanced at the thin, bookish appearance of this new addition to his team. "You sold it?"

Richard thought the squeaking and cracking he heard could have come from the chief's chair or the elderly man's back. "Is this my future?" he wondered. Was the Captain 80? 90? Well, everyone over 50 looked the same.

"Bought the whole act!" smiled Richard. "It looks like I get into the guts of their racket by the end of this week. My interview knocked 'em out!"

Marlow grunted quietly. "Grab that chair. You're from Cranbury, right?"

"Graduated last May and joined their task force in June. They said you needed someone down here who could handle numbers." He talked as he noisily slid the chair over the bumpy floorboards, and sat. "This operation's going to be smooth and clean."

"Well, kid, that crime family you're getting embedded into ain't clean." Marlow leaned back. "Tell me how you did it."

"They gave me a test and I convinced them that I was really good with numbers, so I should be starting as an apprentice accountant sometime soon."

"A test? What kind of a test? I always hated tests."

“It was strange. Sergio, a lieutenant of the Antipasto family, put four blue velvet bags on the desk in a line in front of him. They were small and had a gold-colored tie at the top of each.

Richard continued. “He said 'Boy, how many jewels in each bag?' Well, I shrugged 'cause I couldn't see into the bags. But I knew he was interested in my math abilities, so I asked him, 'Are there a hundred jewels altogether?'”

“Less,” he said.

Marlow, who was always impressed with people who could do math, though he never tried to learn it himself, asked, “And that's when you told him how many were in each bag.”

“Well, no. I kept whittling the total down: 50, 20, 18. Each time he said, 'Less.' Finally, he spit out, 'Enough of this crap.' It was gross. He really did spit! Anyway, then I asked him if all the bags had the same number of jewels, and he said, 'No, they're all different.'”

Marlow nodded and muttered, “And that's when you told him how many were in each.”

“Well, no, it was still too hard a problem, so I asked if I could write down some stuff. He agreed, and I did some calculating on a pad I always carry. I asked, ‘if you tell me what the four numbers multiplied together were, would it help?’ He scribbled on his desk calendar for a short time, and said ‘no’ and then laughed, telling me the product anyway.”

“And that's when you ...”

“That would've been cool, but no. I had one more question and I was nervous. He *had* to answer it.”

“You asked him where the bathroom was.”

“I may have been getting close, but I asked if there was more than one jewel in the bag with the least.”

“And he said?”

“Well, as soon as he answered my question, I told him how many were in each and he told me I could start Friday.”

“Good, good.” Marlow leaned forward and put his elbows on the desk. “I knew you were the man for the job. By the way, how many were in each?”

Richard leaned forward, as Marlow had, but with an infuriating smirk. “I've told you everything you need to know. You can figure it out. But, there IS something I don't know.”

Marlow raised his eyebrows without saying anything, resting his chin on his hands.

Richard's grin faded. “Was there ever really anything in those bags, or was it all hypothetical? Maybe I ought to break into Sergio's office when I'm there and check 'em out.”

Marlow erupted out of his chair and leaned toward Richard. “The hell! You'll get killed. You stick to accounting. That'll give us all we need.”

Richard's eyes drifted toward the ceiling and his face became blank. “I'll need to pick the locks on the doors and desks, and then break the combination on the safe. And a jetpack on the roof in case I gotta' get out fast. And ...”

Marlow sat back down, grabbed the phone and punched the button for his secretary. “Another James Bond wannabe. Lock picking. Ha! Safe cracking. Ha! Ada, get Richard Feynman's file. He's fired! And, let's talk to that Gardner guy.”

- --------

Inspired by: The Scientific American book of Mathematical Puzzles and Diversions, Martin Gardner, Simon and Schuster, 1959, p. 114

Hints

You'll need to write down all the possibilities, including their products. Not as hard as you might think.

Note that Richard couldn't identify the 4 digits even when he heard the product, so eliminate number combinations which gave a product only existing for the one combination.

Of those remaining, his final question allowed him to answer, even though we don't know what Sergio said. If Richard still couldn't have answered at that point, there would have had to have been more questions.

How Squirmfest was Designed

Robert Reid with Michael Dowle, Anthony Steed

Squirmfest is a motif that I (Robert) designed to tessellate the plane. It was intended to be the most intricate and complex tessellation motif that I have created. I outline here how I designed Squirmfest so that tessellation enthusiasts might try to create their own motif(s). The method works for any motif.

I began by drawing a grid structure, a semi-regular tessellation of octagons and squares. I then joined a square and an octagon to form a new unit (Figure 1).

Figure 1: Grid structure – a semi-regular tessellation and new unit

The new unit was then divided into 112 quadrilaterals (squares and rhombuses) in two ways such that the elaborated units were enantiomorphic (mirror images) (Figure 2). The quadrilaterals in both enantiomorphic units were assigned the numbers 1 to 112 (Figure 3). The enantiomorphic units were numbered in black and grey respectively for clarity. The enantiomorphic units were duplicated to give a total of four units that were combined such that the resultant module could tile the plane (Figure 4). Note that both enantiomorphic units appear in two orientations at 180° to each other.

Figure 2: Enantiomorphic elaborated units containing 112 quadrilaterals

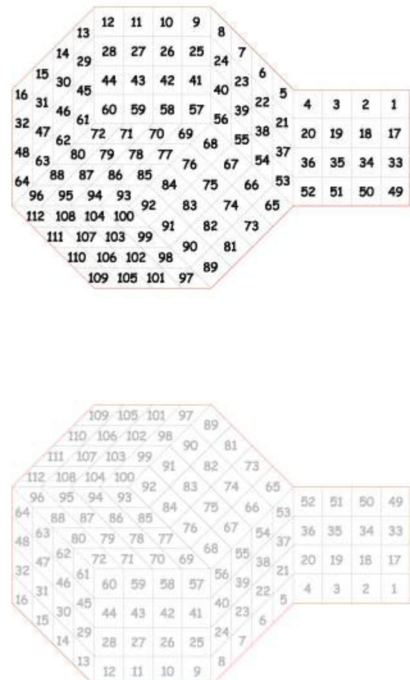


Figure 3: Numbered enantiomorphic elaborated units

To create the Squirmfest motif I chose 112 (the number of quadrilaterals in an elaborated unit) edgewise-connected quadrilaterals (Figure 5) according to the following criteria:

- the quadrilaterals did not have to reside within the same elaborated unit;
- when moving across different elaborated units no numbered quadrilateral was used a second (or more) time(s) - I used a checklist of the numbers 1 through to 112 to prevent this occurrence;
- all 112 areas numbered 1 through to 112 were used; no enclosed areas were created.

The resultant four Squirmfest motifs for tiling the plane were drawn. Note the two enantiomorphs each in two orientations (enantiomer 1 – red and green; enantiomer 2 – blue and yellow) (Figure 6). Note how the numbering in all four motifs is similar. The red, blue, yellow and green motifs were then combined to provide the module ready to tile the plane (Figure 7). Note how the module sits on the modified grid. 18 modules were used in the illustrated Squirmfest tessellation (Figure 8).

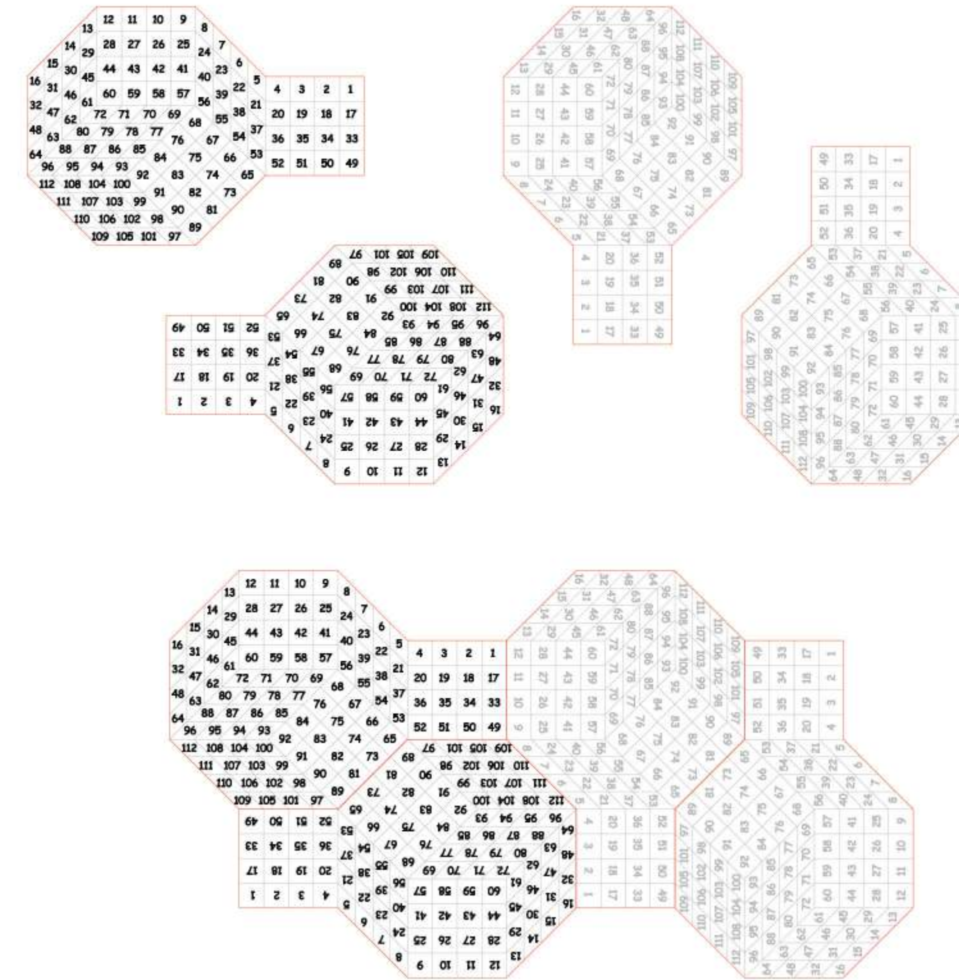


Figure 4: Combination of four units into a module

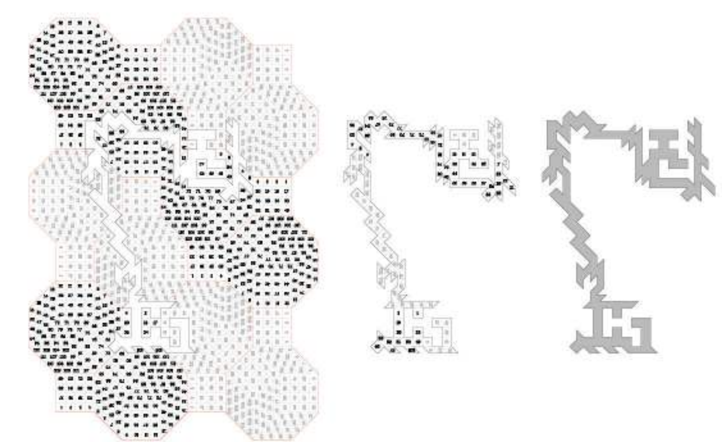


Figure 5: Creation of the Squirmfest motif

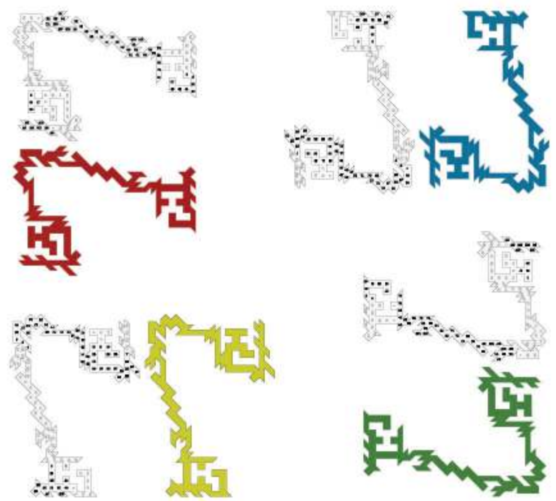


Figure 6: Four Squirmfest motifs – two enantiomorphs each in two orientations

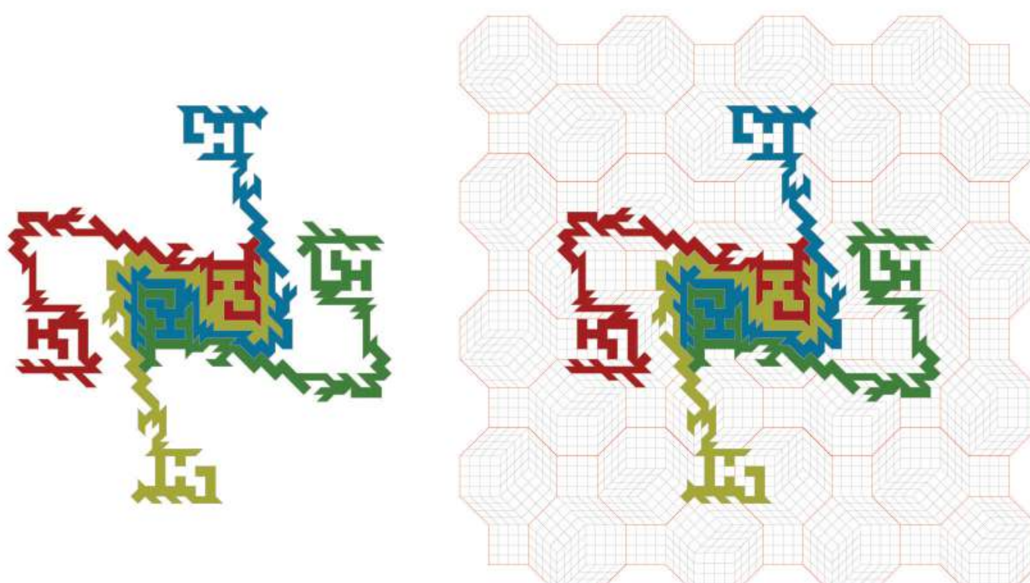


Figure 7: Squirmfest module

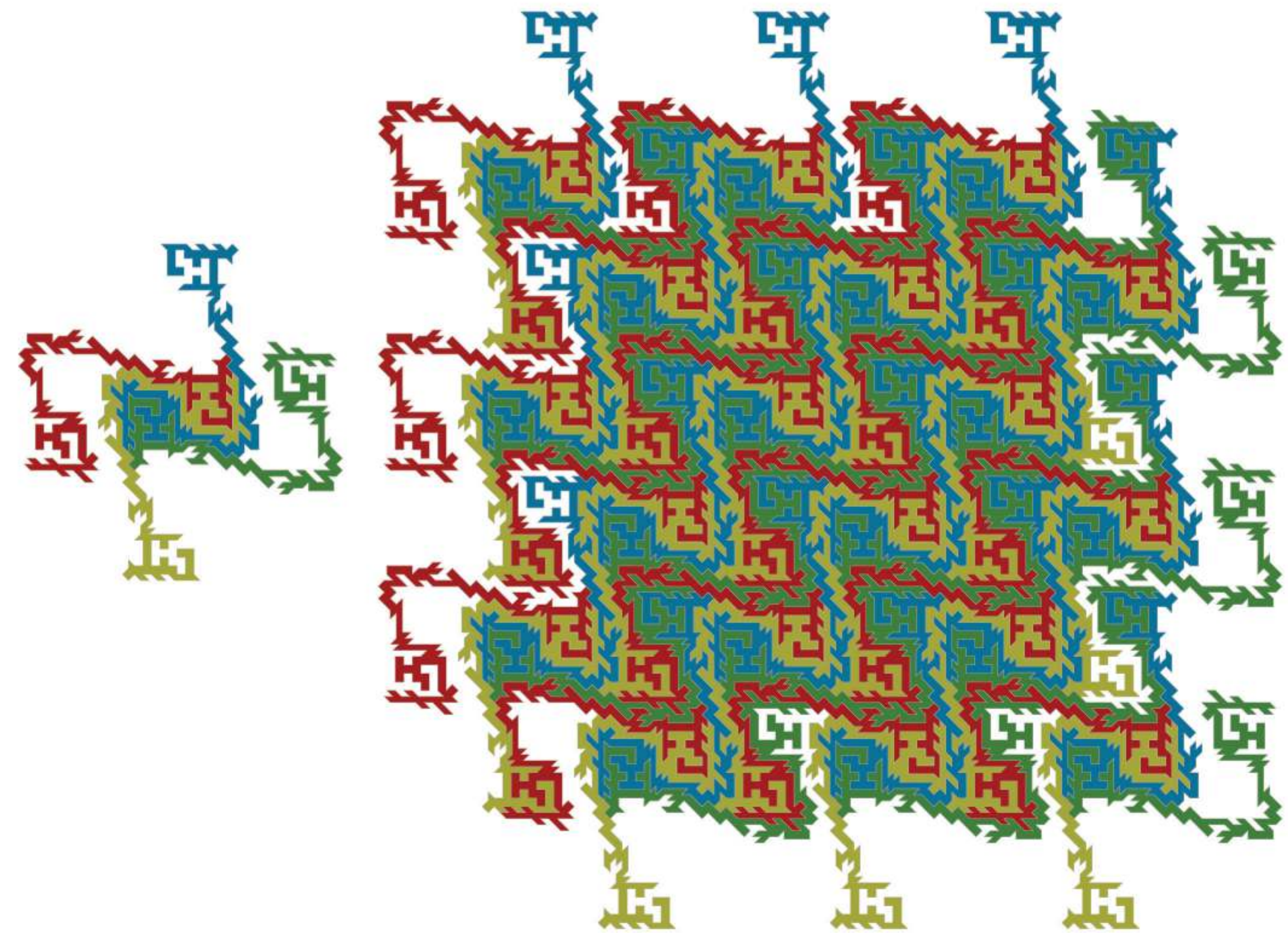


Figure 8: Squirmfest tessellation

The Martian Mayor Problem

T. Arthur Terlep, Purdue University
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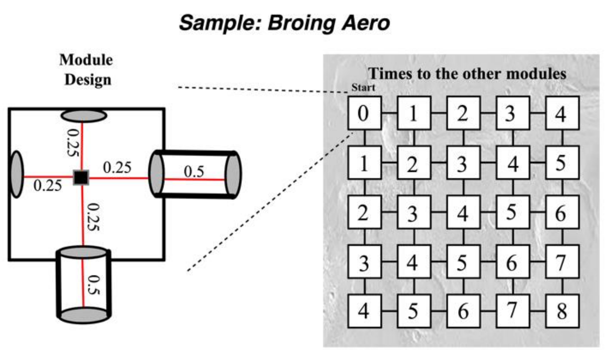
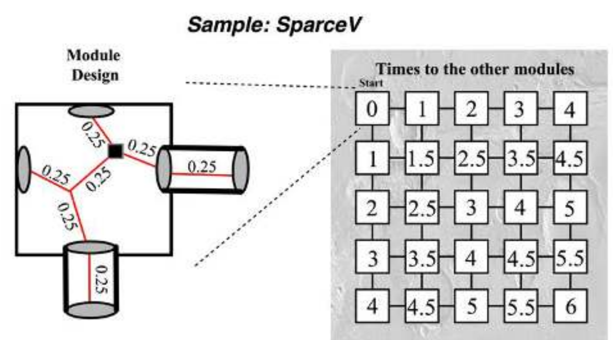
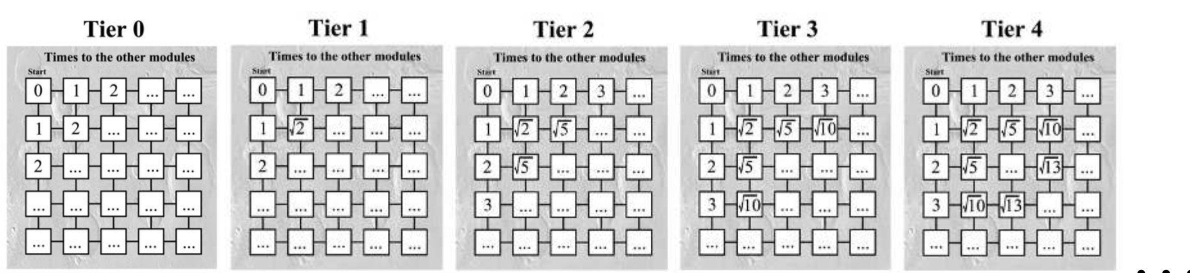
G4G14

Mayor Martin Martian oversees the contract bids for the design of the huge square module used in the large-scale colonization effort which will tile the Martian landscape. Contractors decide the layout and speed of bidirectional trolley paths and the location of the command center, which must all be constructed the same inside every module. Assume that all trolley intersections are seamless interchanges. Two engineering firms have already submitted designs for the contract (for the sake of example). On the left of each sample is the module design with the travel time along each trolley segment. On the right is the Timing Table. These measure the minimum travel time in hours between the top left module and the other modules. All paths must start and stop on a command center, but do not need to visit intermediate command centers along the way. Each module has two connecting tunnels which permit ONLY a single trolley path. Paths may NOT bridge over each other.





Additionally, a recent Psychophysics research shows humans prefer more "Euclidean travel times" so there is a $1 million bonus for each Tier above 0 achieved in the final design. Each Tier is shown in the Timing Tables below. There is no limit on the number or speed of trolley paths inside the module.



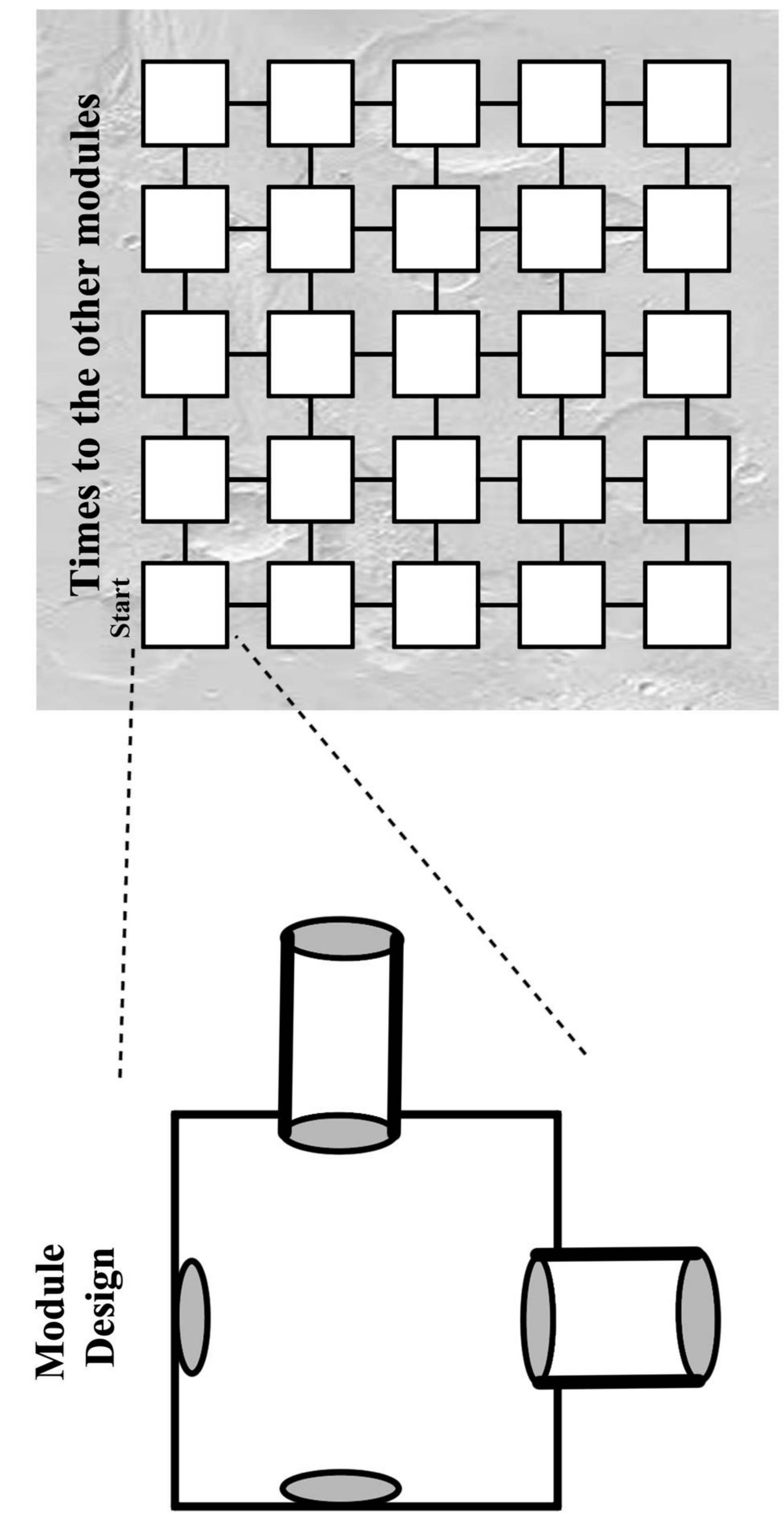
You work for a contracting company submitting a bid.
1. What is the module design for the best Tier you can achieve with a single tunnel between modules?

Unhappy with progress, the city council has intervened and allowed for two tunnels between adjoining modules.
2. What is the module design for the best Tier you can achieve with two parallel tunnels between adjoining modules? Three or more parallel tunnels?
3. What is the relationship between the number of tunnels and the highest achievable Tier?

A SparceV lobbyist campaigns to allow trolley paths inside the module to bridge over each other (Non-planar paths)
4. What has the lobbyist discovered? What is the module design of the best Tier for 1, 2, and 3 tunnels?

Big brain company Goobr has proposed some alternative shapes and the city council is changing the design again…
5. What do these modules look like for other tiling shapes (triangles, hexagons, and irregular tiles)?

Please send your solutions to Art Terlep, taterlep@gmail.com



Solving Rep-tile by Computers: Performance of Solvers and Analyses of Solutions

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Kakeru Nakamura§ Masaaki Nishino¶ Masahiko Sakai* Ryuhei Uehara§
Yushi Uno‖ Norihito Yasuda¶

October 12, 2021

Abstract

A *rep-tile* is a polygon that can be dissected into smaller copies (of the same size) of the original polygon. A *polyomino* is a polygon that is formed by joining one or more unit squares edge to edge. These two notions were first introduced and investigated by Solomon W. Golomb in the 1950s and popularized by Martin Gardner in the 1960s. Since then, dozens of studies have been made in communities of recreational mathematics and puzzles. In this study, we first focus on the specific rep-tiles that have been investigated in these communities. Since the notion of rep-tiles is so simple that can be formulated mathematically in a natural way, we can apply a representative puzzle solver, a MIP solver, and SAT-based solvers for solving the rep-tile problem in common. In comparing their performance, we can conclude that the puzzle solver is the weakest while the SAT-based solvers are the strongest in the context of simple puzzle solving. We then turn to analyses of the specific rep-tiles. Using some properties of the rep-tile patterns found by a solver, we can complete analyses of specific rep-tiles up to certain sizes. That is, up to certain sizes, we can determine the existence of solutions, clarify the number of the solutions, or we can enumerate all the solutions for each size. In the last case, we find new series of solutions for the rep-tiles which have never been found in the communities.

1 Introduction

In some games like Tetris, polygons obtained by joining unit squares edge to edge are used as their pieces. These polygons are called polyominoes, and they have been used in popular puzzles since at least 1907. Solomon W. Golomb introduced the name polyomino in 1953 and was widely investigated [1]. It was popularized in the 1960s by the famous column in *Scientific American* written by Martin Gardner [2].

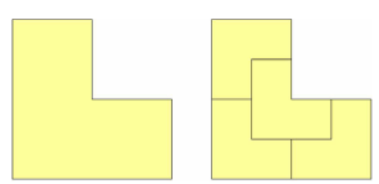

Figure 1: An example of a rep-tile of rep-4.

Golomb is also known as an inventor of the notion of rep-tile. A polygon P is called rep-tile if it can be dissected into smaller copies of P. Especially, if P can be dissected into n copies, it is said to be rep-n. An example of a rep-tile of rep-4 is given in Figure 1. We can observe that each of 4 copies can be dissected into 4 smaller copies, which give us rep-16. That is, a rep-tile of rep-n is also rep-n^i for any positive integer $i = 1, 2, \ldots$. We also extend the rep-tile of rep-n by tiling n copies to make a larger pattern. That is, we can tile the plane by repeating this process. It is known that some rep-tile can be used to generate acyclic tiling (i.e., the tiling pattern cannot be identical by shifting and rotation). Both cyclic and acyclic tilings have been well investigated since they have applications to chemistry, especially, crystallography [3]. From the viewpoints of mathematics and art, the notion of rep-tile is popular as we can obtain tiling of the plane with the same shapes of different sizes by replacing a part of the rep-tiles by their copies recursively.

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‖Osaka Prefecture University, Japan. uno@cs.osakafu-u.ac.jp

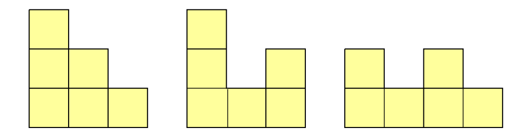


Figure 2: The 6-ominoes of *stair-shape*, *J-shape*, and *F-shape*

Gardner introduced the polyomino rep-tiles in [3]. Precisely, he introduced three 6-ominoes (polyominoes formed by 6 unit squares) in Figure 2 as rep-tiles of rep-144. When the article [3] was written, the minimum number of dissections for these three rep-tiles was conjectured as 144. Namely, they are the rep-tiles of rep-144, and not rep-k for any $1 < k < 144$. However, they have been found out that the left *stair-shape* is a rep-tile of rep-121, the central *J-shape* is a rep-tile of rep-36, and the right *F-shape* is a rep-tile of rep-64 [3, 4].

Polyomino rep-tiles have a long history mainly in the contexts of puzzles and recreational mathematics. They have been investigated since the 1950s, however, they have relied on discoveries by hand. In fact, there are many constructive solutions for these puzzles on the web page [4] However, these puzzles have not yet "solved" in the strict sense since any nonexistent results for these cases have not yet be given.

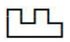
In this research, we first experiment on these three polyominoes in Figure 2 and check if they are rep-n for each n by the representative approaches by a computer. Since the notion of a rep-tile is a quite simple puzzle, we can represent the conditions of a rep-tile in several different natural ways in the terms of representative problem solvers. Therefore, we can compare the performance of the different problem solvers using such a simple puzzle as a common problem. We use the following three different approaches to solving the rep-tiles by a computer.

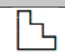
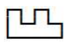
Puzzle solver and implementation based on dancing links: Nowadays, most puzzle designers use a free puzzle solver. It is based on a data structure called dancing links proposed by Knuth. It is said that dancing links is the data structure that allows us to perform backtracking efficiently, and hence it is suitable to analyze puzzles. Although we do not know the details of the implementation of the free puzzle solver, we also independently implemented two algorithms; one uses dancing links, and the other uses dancing links with ZDD to make it faster.

MIP solver: When we formalize the solutions of a rep-tile by constraint integer programming, we can solve it by mixed integer programming (MIP) solvers. The conditions of a rep-tile can be formalized in a relatively simple integer programming (IP), and we can decide if the rep-tile has a solution if and only if the corresponding instance in the form of the IP is feasible. Since each feasible solution corresponds to a solution, the number of feasible solutions also gives the number of solutions of the rep-tile. In this formulation, the feasibility is the issue and hence the optimization term in the MIP solver is redundant.

SAT-based solver: Most instances of the integer programming can be solved by SAT-based solvers with some modifications of constraints. It is the case for the conditions of a rep-tile, and hence the IP formulation can be translated to the constraints of the SAT-based solvers.

In summary, the puzzle solver and programs based on dancing links, even if we use ZDD, cannot solve rep-tiles of rep-n for large n. However, this fact does not mean the limit of using a computer. The MIP solver can solve rep-tiles of rep-n for larger n than the puzzle solvers. Moreover, we found out that the

k	1	2	3	4	5	6	7	8	9	10	11
	1	0	0	0	0	0	0	0	0	0	328582628812951388l6
	1	0	0	0	0	262144	0	0	0	0	0
	1	0	0	0	0	0	0	1358954496	51539607552	0	0

k	12	13	14	15	16
	751374255349863353l870412820	421105971327597731222250323968	0	0	0
	545409716939029673955819520	0	0	0	0
	693242756013012824879005696	3658830332096120778961977344	0	> 0	> 0

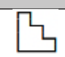
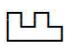
k	17	18	19	20	21	22	23	24	25
	0	0	?	?	?	?	> 0	> 0	> 0
	0	> 0	0	0	0	0	0	> 0	0
	> 0	?	> 0	> 0	> 0	?	> 0	> 0	> 0

Table 1: The number of distinct dissections of k^2-omino rep-tiles, where each number indicates the number of solutions, where 0 means no solution, > 0 means at least one solution, and ? means unknown.

SAT-based solvers can solve much larger sizes than the MIP solver. These results were contrary to our expectations.

By using a model counting method with a SAT-based solver, we succeeded to count the number of solutions of rep-tiles of certain sizes, which are bigger than the previously known results. Our results are summarized in Table 1. (As we will describe later, there exist n-omino rep-tiles only when $n = k^2$ for some positive integer k. Therefore, we will consider k^2-omino rep-tiles for $k = 1, 2, \ldots$.)

By examining in detail the number of solutions and the specific individual solutions, we obtain two major new results regarding rep-tiles.

Each 0 in Table 1 indicates that there is no rep-tile of rep-k^2 for the corresponding 6-omino. Since the previously known results of rep-tiles only indicate the existence of a solution constructively, it remains open whether there is a solution for other sizes. In this paper, we show for the first time that there is no solution up to a certain size. It was conjectured that these three rep-tiles of rep-144 were the minimum size in [3], and then gradually, smaller solutions were shown constructively. However, it has never been proved that they are the minimum number. Our results in Table 1 reveal for the first time that they are all the minimum rep-tiles. They put an end to the history of exploration of these rep-tiles for more than 50 years.

As for the size in which solutions exist, we succeed in completely characterizing some of the solutions by analyzing the number of solutions and patterns of these solutions. They contain whole new types of solutions that are not included in previously known constructive solutions. We also succeed in constructing solutions with completely different characteristics from the known solutions by combining a constructive method and a search using these new types of solutions as clues. By developing these new types of solutions, it may be possible to find completely new solutions even for sizes that are previously expected to have no solution.

2 Preliminaries

A *polyomino* is a simple polygon that can be obtained by joining unit squares edge by edge. All polygons in this paper are polyominoes. For an integer s, a polyomino of area s is called an s-*omino*. A simple polygon P is a *rep-tile* of *rep-n* if P can be dissected into n congruent polygons similar to P.

In this paper, we focus on polyomino rep-tiles. Then the following theorem is important.

Theorem 1 *When a polyomino P is a rep-tile of rep-n, n is a square number. That is, there exists a natural number k such that $n = k^2$.*

Proof Let P be a t-omino. That is, P consists of t unit squares. By assumption, P can be dissected to n copies of P', where P' is similar to P. Then, since a unit square has an edge of length 1, the corresponding square of P' has an edge of length $1/\sqrt{n}$. Let ℓ be the length of a shortest edge e of the polyomino P. Then, ℓ is an integer and ℓ should be a multiple of $1/\sqrt{n}$ since this edge e is formed by tiling P'. Therefore, $\sqrt{n}$ should be an integer, and hence n is a square number. ∎

By Theorem 1, a polyomino P cannot be a rep-tile of rep-n when n is not a square number. Therefore, we assume that $n = k^2$ for some positive integer k without loss of generality. In order to compare to the previous results, we focus on the three 6-ominoes shown in Figure 2 in this paper. We call each of them *stair-shape*, *J-shape*, and *F-shape*, respectively.

Among these three 6-ominoes, the J-shape and the F-shape are concave, and hence the concave part should be filled by the other piece to construct a rep-tile. Precisely, a polyomino P is *concave* if there exists a unit square not belonging to P but it shares three edges with P. We call this square *concave square* of P.

In this research, we solve the polyomino rep-tile problem for the three 6-ominoes by some problem solvers. When we use MIP solver or SAT-based solvers, we have to describe the constraints of the rep-tile problem. Here we give the common way for the representation.

As a simple example, we consider a domino (or 2-omino) P of rep-4. In this case, since $4 = 2^2$ is the square number of $k = 2$, we consider P as an 8-omino of size 4×2 by scaling 2 and fill P by 4 dominoes of size 2×1. We first assign a unique number to each unit square of P. We let

0	1	2	3
4	5	6	7

for example. When we tile 4 dominoes on the 8-omino P, a binary variable $A(i, j)$ using the numbers of unit squares indicates a way of each domino. To make the representation unique, we assume that $i < j$. For example, when $A(0, 1) = 1$, it means that a domino covers the unit squares 0 and 1. For this P, we use 10 binary variables ($A(0, 1)$, $A(1, 2)$, $A(2, 3)$, $A(4, 5)$, $A(5, 6)$, $A(6, 7)$, $A(0, 4)$, $A(1, 5)$, $A(2, 6)$, $A(3, 7)$) to represent if a domino covers the corresponding unit squares.

Next, we introduce constraints for each unit square. Precisely, since each unit square i should be covered by just one domino, we have the following constraints.

Constraint for the square 0 : $A(0, 1) + A(0, 4) = 1$

Constraint for the square 1 : $A(0, 1) + A(1, 2) + A(1, 5) = 1$

Constraint for the square 2 : $A(1, 2) + A(2, 3) + A(2, 6) = 1$

Constraint for the square 3 : $A(2, 3) + A(3, 7) = 1$

Constraint for the square 4 : $A(4, 5) + A(0, 4) = 1$

Constraint for the square 5 : $A(4, 5) + A(5, 6) + A(1, 5) = 1$

Constraint for the square 6 : $A(5, 6) + A(6, 7) + A(2, 6) = 1$

Constraint for the square 7 : $A(6, 7) + A(3, 7) = 1$

It is clear that P is a rep-tile of rep-4 if and only if there is a solution that satisfies these eight constraints.

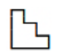
In this paper, we wrote programs that generate the declarations of the binary variables and the corresponding constraints for each combination of 6-ominoes stair-shape, J-shape, or F-shape, and a square number $n = k^2$.

3 Comparisons of Solvers

As representative problem solvers, we chose BurrTools as a puzzle solver, SCIP as a MIP solver, and clingo and NaPS as SAT-based solvers. For each of the three rep-tiles, we list their running time for solving the rep-tile. The details and resources of the solvers follow them. Tables 2, 3, 4 summarize the running times of solvers for each rep-tile. In the tables, DLX indicates the algorithm based on dancing links, DLZ indicates the algorithm based on dancing links with ZDD, which are implemented by ourselves to compare with BurrTools. We omit the cases $k < 6$ since they are too short, and each number represents seconds. The symbol ? means timeout in this case. We set the time limit for each solver as 10 minutes (600 seconds) in DLX/DLZ, 12 hours (43200 seconds) in clingo, and 2 days (172800 seconds) in NaPS. The entry OF in DLZ means "overflow of cache". After each k, we put ✓, ×, and

? which mean “there exists a solution”, “there exists no solution”, and “we do not know if there is a solution or not,” respectively.

k (Solution?)	6(×)	7(×)	8(×)	9(×)	10(×)	11(✓)	12(✓)	13(✓)	14(×)	15(×)
BurrTools 0.6.3	< 1	2	5760	?	?	?	?	?	?	?
DLX(1st solution)	< 1	15	?	?	?	?	< 1	?	?	?
DLX(all solutions)	?	?	?	?	?	?	?	?	?	?
DLZ(1st solution)	< 1	< 1	< 1	391	OF	OF	< 1	OF	OF	OF
DLZ(all solutions)	?	?	?	?	?	?	OF	OF	?	?
SCIP 7.0.2	1	1	1	1	56	42	7	120	?	?
clingo 5.4.0	< 1	< 1	< 1	< 1	< 1	1	2	2	8	?
NaPS 1.02b2	< 1	< 1	< 1	< 1	< 1	< 1	< 1	1	17	6388
k (Solution?)	16(×)	17(×)	18(×)	19(?)	20(?)	21(?)	22(?)	23(✓)	24(✓)	25(✓)
clingo 5.4.0	?	2946	?	?	?	?	?	8911	26973	?
NaPS 1.02b2	(421700)	1163	12530	?	?	?	?	529	1415	1744

Table 2: Time (sec.) for deciding if  is a rep-tile of rep-k^2 (NaPS finishes its computation after the time limit when $k = 16$)

k (Solution?)	6(✓)	7(×)	8(×)	9(×)	10(×)	11(×)	12(✓)	13(×)	14(×)	15(×)
BurrTools 0.6.3	6	< 1	< 1	?	?	?	?	?	?	?
DLX(1st solution)	< 1	< 1	?	?	?	?	< 1	?	?	?
DLX(all solutions)	< 1	?	?	?	?	?	?	?	?	?
DLZ(1st solution)	< 1	< 1	< 1	< 1	< 1	241	< 1	146	OF	OF
DLZ(all solutions)	< 1	?	?	?	?	?	6	?	?	?
SCIP 7.0.2	1	1	1	5	9	14	69	4	6	19800
clingo 5.4.0	< 1	< 1	< 1	1	2	5	4	2	4	5
NaPS 1.02b2	< 1	< 1	< 1	< 1	1	2	2	7	52	116
k (Solution?)	16(×)	17(×)	18(✓)	19(×)	20(×)	21(×)	22(×)	23(×)	24(✓)	25(×)
clingo 5.4.0	6	11	9	5336	41489	?	?	?	1454	?
NaPS 1.02b2	208	282	113	1531	116400	?	?	?	1675	?

Table 3: Time (sec.) for deciding if  is a rep-tile of rep-k^2

Comparing to the DLX based on just dancing links, BurrTools implements some more tricks. The DLZ, which uses not only dancing links but also ZDD, performs more efficiently than DLX, however, it causes memory overflow when the search space becomes larger. Comparing to the algorithms based on dancing links, the MIP solver SCIP can deal with a larger scale. We note that we do not need the optimization function of the MIP solver in rep-tile. When we use SAT-based solvers clingo and NaPS, the range that can handle is much wider than the other problem solvers.

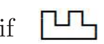
The details of each experiment are described below, however, there are differences in resources depending on problem solvers. This is because the authors split up to perform experiments that was good at each tool. The difference in computation results due to the difference in resources is considered to be tens to hundreds of times, however, considering the scale of the problem that increases exponentially and the actual computation results in Tables 2, 3, and 4, it can be seen that the differences of these constant factors do not affect our conclusion. The following are the details for each experimental environment.

3.1 Puzzle solvers

BurrTools 0.6.3[1] is widely recognized as the standard puzzle solver in the puzzle society. It supports a variety of grids and also supports 2D and 3D for puzzles that ask to pack a given set of pieces into

[1] http://burrtools.sourceforge.net/

k (Solution?)	6(×)	7(×)	8(✓)	9(✓)	10(×)	11(×)	12(✓)	13(✓)	14(×)	15(✓)
BurrTools 0.6.3	< 1	960	172800	?	?	?	?	?	?	?
DLX(1st solution)	< 1	< 1	< 1	< 1	?	?	< 1	?	?	?
DLX(all solutions)	?	?	?	?	?	?	?	?	?	?
DLZ(1st solution)	< 1	< 1	< 1	< 1	< 1	102	< 1	20	?	?
DLZ(all solutions)	?	?	< 1	< 1	?	?	OF	OF	?	?
SCIP 7.0.2	1	2	43	13	11	259200	?	?	?	?
clingo 5.4.0	< 1	< 1	< 1	1	3	4	11	37	97	372
NaPS 1.02b2	< 1	< 1	< 1	< 1	2	3	10	14	671	688
k (Solution?)	16(✓)	17(✓)	18(?)	19(✓)	20(✓)	21(✓)	22(?)	23(✓)	24(✓)	25(✓)
clingo 5.4.0	244	134	?	18022	6498	?	?	?	?	?
NaPS 1.02b2	316	505	?	7455	6249	8485	?	47550	131900	146200

Table 4: Time (sec.) for deciding if  is a rep-tile of rep-k^2

a given frame (without overlapping or gaps). According to the web page of BurrTools, it is based on the data structure dancing links proposed by Knuth, who wrote a 270-page textbook [5]. Dancing links is a data structure for efficiently performing backtracking in a tree search by depth-first search. In the literature [5], many examples are taken from famous puzzles as applications of backtracking in search trees. In fact, the polyomino packing puzzle, which is essentially the same as the rep-tile, is also taken up in detail as an example. In our experiments, the machine used has an Intel Core i5-7300U (2.60GHz) CPU and 8GB of RAM. It is the limit of analysis for $k = 8$, namely, the rep-tile of rep-64 in each pattern.

BurrTools does various tunings internally, however, the details are not public. For comparison, we first implemented using dancing links as they are. The machine used has a CPU of Ryzen 7 5800X (3.8GHz) and 64GB of RAM. The C program for the experiment used DLX1[2] developed by Knuth. When using DLX1, it turns out that $k = 12$ is the limit in terms of finding a solution, and $k = 6$ is difficult in terms of finding all solutions. Next, we tried to speed up the search by combining dancing links with ZDD. The C program for the experiment used DLX6[3] developed by Knuth. The word ZDD is an abbreviation for Zero-suppressed Binary Decision Diagram, and it is a data structure that shares subtree structures that appear in common in the binary decision tree. In particular, the memory efficiency is further improved compared to the normal BDD by not maintaining the path when the result becomes 0 (see [7] for details). If ZDD is used in a tree search like our problem, since it is not necessary to repeatedly search the already searched subtree, a significant speedup can be expected. On the other hand, it is necessary to store all the subtrees once searched in the cache, and hence the memory efficiency is worse than the depth-first search tree. By speeding up using ZDD, it is possible to achieve up to $k = 13$ in the sense of finding a solution, and $k = 9$ for the F-shape and $k = 12$ for the J-shape in the sense of finding all the solutions. However, when the scale was larger than that, the search could not be completed due to lack of memory.

3.2 MIP solver

As the MIP solver, SCIP 7.0.2 [4] was used in this research. The way of modeling is as introduced in Section 2. SCIP requires a term for optimization, however, it is redundant in our model. Hence, we minimize the sum of binary variables as a dummy. Whenever it is feasible, the result comes to $n = k^2$, so it acts as a double check for the feasible solution.

The machine used in the experiment has an Intel Core i5-7300U (2.60GHz) CPU and 8GB of RAM. Although the results a bit vary, it can be seen that the solvable range is wider than when BurrTools is used.

[2] See https://www-cs-faculty.stanford.edu/~knuth/programs.html and https://www-cs-faculty.stanford.edu/~knuth/programs/dlx1.w for the details.

[3] https://www-cs-faculty.stanford.edu/~knuth/programs/dlx6.w

[4] https://www.scipopt.org/

3.3 SAT-based solvers

Some SAT-based solvers support Pseudo Boolean Constraints (PBs) (see [6] for details). All the constraints used in the above MIP solver are within the range of PB except for the optimization term. Here, the optimization term in the MIP solver was redundant information when finding solutions of the rep tile. Therefore, when the optimization term is deleted from the constraint descriptions used in the above MIP solver, it can be solved by the SAT-based solvers that can handle PBs as they are.

In this research, we used two typical SAT-based solvers for deciding satisfiability; clingo 5.4.0 [5] and NaPS 1.02b2 [6]. The machine used to run clingo has an Intel Core i7 (3.2GHz) CPU and 64GB of memory, and the machine used to run NaPS has a Core i3 (3.8GHz) CPU and 64GB of memory. Each computation time corresponds to the time for finding the first solution in the other solvers. Even considering the differences among the execution environments, it can be concluded that the range that can be solved by the SAT-based solvers is dramatically expanded compared to the puzzle solver and the MIP solver.

3.4 How to count the number of solutions

From the experiments, it was found that the best method for determining the existence of the solution is to use the SAT-based solvers. The SAT-based solvers used in Section 3.3 have the function of finding all solutions in addition to determining whether or not it is satisfiable. However, it is not practical since it will take time due to the large number of solutions. On the other hand, the projected model counting solver GPMC[7] cannot find a solution for given constraints in CNF, however, the number of solutions can be found at high speed.

Therefore, in order to compute the number of solutions, we first determine the satisfiability using NaPS, next convert the constraints described in PB to the CNF using the conversion function of NaPS if it is satisfiable. Then the number of models was counted by GPMC. (To be more precise, when PBs are converted to CNF, variables other than the binary variable of interest are also generated. Therefore, the GPMC projection model counting function is used to count only the number of satisfiable assignments to the variable of interest. We can count the number of satisfiable solutions by this way.)

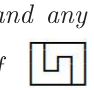
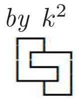
Table 1 summarizes the number of solutions obtained by combining NaPS and GPMC in this way. The entry written as > 0 in the table is the entry confirmed that the solution exists using NaPS, and the entry that specifically describes the number of solutions is the entry that was successfully counted by GPMC. The ? mark indicates that any solution could not be found after running NaPS for 2 days. Here, for the $k = 16$ in stair-shape, a solution was found when the time limit was exceeded.

4 Analysis and New Solutions

As shown in Section 3, through this research, we were able to compute the number of solutions of rep-tile solutions up to a previously unknown size. Specifically, in each case, the existence of solutions was determined by NaPS, and the number of solutions was counted by GPMC. However, although the total number of solutions can be found with this method, the details of the solutions are not clear. In this section, we observe the solutions by NaPS and the number of solutions by GPMC, referring to the known results, and clarify the details of solutions for some k. As a result, we find new solutions that were not included in the known results at all. We will look at this in detail for each 6-omino.

4.1 J-shape 6-omino

The following property is useful for analysis of J-shape 6-omino (hereafter, we assume $k > 1$ to simplify):

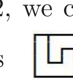
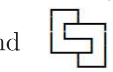
Lemma 2 *Let k be an integer such that J-shape 6-omino is a rep-tile of rep-k^2. Then k^2 is an even number and any tiling by k^2 copies of J-shape can be dissected into $k^2/2$ 12-ominoes such that they consist of*  *and*  *(or their mirror images).*

[5] https://potassco.org/clingo/
[6] https://www.trs.cm.is.nagoya-u.ac.jp/projects/NaPS/
[7] https://www.trs.cm.is.nagoya-u.ac.jp/projects/PMC/

Proof A J-shape piece P is concave. That is, it has a unit square not belonging to P but sharing three edges of P. To cover this unit square by the other J-shape, we have only two ways shown above. This implies the lemma. ∎

We obtain a corollary by Lemma 2.

Corollary 3 *For any odd number $n > 1$, a J-shape 6-omino is not a rep-tile of rep-n. Therefore, for any odd number $k > 1$, a J-shape 6-omino is not a rep-tile of rep-k^2.*

By Theorem 1 and Corollary 3, it is sufficient to check whether a J-shape 6-omino is a rep-tile of rep-k^2 only for even k. Moreover, by Lemma 2, we can decide if a J-shape 6-omino is a rep-tile by checking of tiling using only two 12-omino pieces  and . Using this method, we can complete the computation for k larger than the experiments in Section 3. By combining the arguments with the results in Section 3, we obtain the following theorem for the J-shape 6-omino:

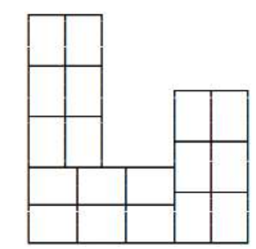


Figure 3: Construction of tiling based on rectangles of size 3×4 for $k = 6$

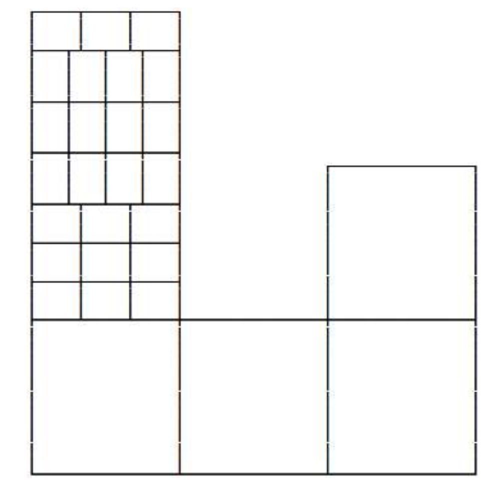
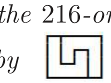
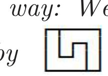
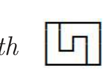
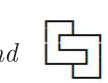
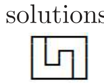
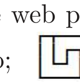
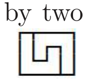


Figure 4: Part of construction of tiling based on rectangles of size 3×4 for $k = 12$

Theorem 4 *For a rep-tile of the J-shape 6-omino of rep-k^2, we have the following:*
(0) There exists no rep-tile of rep-k^2 for an odd number k (except $k = 1$). There exists no rep-tile of rep-k^2 for $k = 2, 4, 8, 10, 14, 16, 20, 22$.
(1) Case $k = 6$: All solutions can be obtained by the following way: We first dissect the 216-omino P into 18 rectangles of size 3×4 as shown in Figure 3 and then replace each rectangle by  *or its mirror image.*
(2) Case $k = 12$: All solutions can be obtained by the following way: We first dissect the 864-omino P into 72 rectangles of size 3×4 and then replace each rectangle by  *or its mirror image.*
(3) Case $k = 18, 24$: There are some solutions that contain both  *and* .

Proof (0) We can obtain the results for odd k by Corollary 3. The search results by SAT-based solvers in Table 3 give us the results for $k \leq 20$. By Lemma 2, we perform the search of tiling by copies of two 12-ominoes for larger k. Using NaPS, we confirmed that there is no rep-tile of rep-k^2 for $k = 22$.
(1) Case $k = 6$: The known solutions for the J-shape 6-omino on the web page [4] are based on the arrangement of the rectangle . In fact, when $k = 6$, the pattern in which 18 rectangles are arranged (Figure 3) is shown on the web page. There are two ways to dissect each rectangle to a pair of two copies of the J-shape 6-omino;  or its mirror image. When $k = 6$, the number 262144 of solutions matches $2^{18} = 262144$. That is, in the case of $k = 6$, there are at least 2^{18} solutions based on

the dissection into the rectangles in Figure 3, which is equal to the number of solutions actually counted by GPMC. Since they match, we can guarantee that no other solution exists.

(2) When $k = 12$, the number of solutions is 5454097169390296739558195200. This number is much larger than 2^{72}, which is obtained by the same dissection of the case $k = 6$. The reason can be expressed as follows. We first consider a square corresponding to the unit square of the J-shape polyomino P. In the rep-tile for $k = 12$, the square is of size 12×12. Then we can tile this square by tiling 12 rectangles of size 3×4 in vertical or horizontal. (We note that we have no such a choice in Figure 3, and the dissection is uniquely determined.) Therefore, we have to consider the number of ways of tiling of rectangles in vertical or horizontal. Moreover, when we consider a large rectangle obtained by joining these squares of size 12×12, there are variants of tiling of rectangles of size 3×4. A concrete example is shown in Figure 4. In this example, the rectangle of size 12×24 in P is dissected into rectangles of size 12×3, 12×12, and 12×9. It is not easy to count the number of distinct dissections of P into rectangles of size 3×4. Therefore, we first count the number of dissections of P for $k = 12$ into rectangles of size 3×4 (and 4×3) by GPMC, which finishes soon. As a result, the number of ways of dissections is 115495. Here, we can confirm that $5454097169390296739558195200 = 115495 \times 2^{72}$. Therefore, every rep-tile for $k = 12$ can be obtained by two steps; first, dissect P into rectangles of size 3×4 and 4×3, and then replace each of them by  or its mirror image.

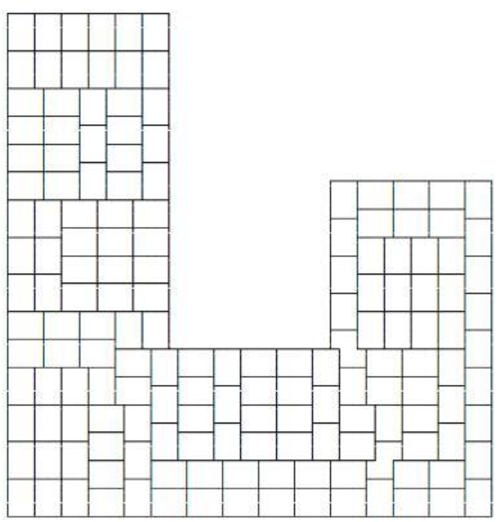
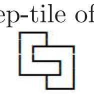


Figure 5: A rep-tile of rep-18^2 of J-shape 6-omino that contains 

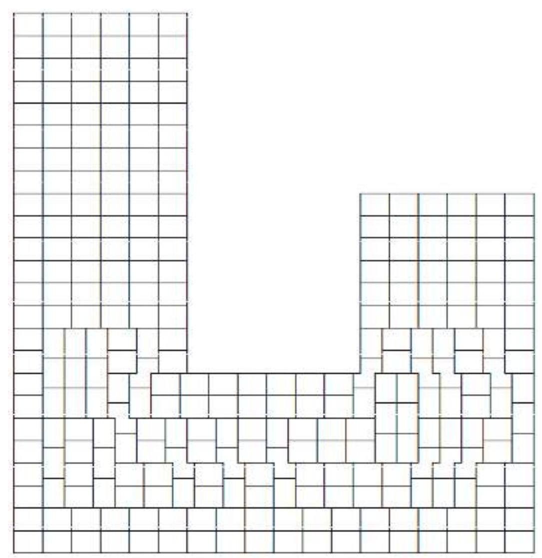
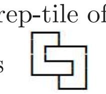
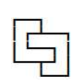
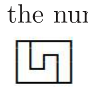
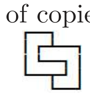
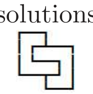
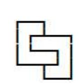
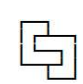
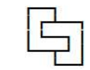
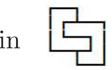
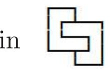
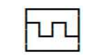
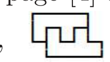
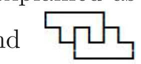
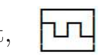


Figure 6: A rep-tile of rep-24^2 of J-shape 6-omino that contains 

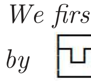
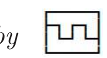
(3) By Lemma 2, we can decide if there is a solution that uses  in a tiling by J-shape 6-omino by searching using two types of 12-ominoes. Moreover, when we specify the range of the number of copies of each of two 12-ominoes, we can decide if there is a solution that contains both  and . As a result, we found that there were such solutions for $k = 18, 24$; see Figure 5 and Figure 6. ∎

We note that the solutions that contain  are new solutions not included in previously known results. So far, in the case $k = 18$, there are solutions that contain x copies of  for every even number x from 2 to 46. There is no such solution when $x \geq 47$. That is, all solutions containing  we found have even number of pairs of this form. It is not known the details for $k = 18$: For example, the number of solutions in the case $k = 18$, whether there exists a solution that contains an odd number copies of , and how many solutions that contain  are not known. We conjecture that there are solutions that contain  for $k > 24$.

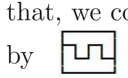
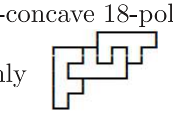
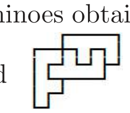
4.2 F-shape 6-omino

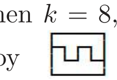
The known rep-tiles of the F-shape 6-omino are a bit complicated, however, the solutions posted on the web page [4] are explained as follows: We first combine two copies of the F-shape 6-omino to form , , and , then next arrange them appropriately, and finally place one copy of the F-shape 6-omino if necessary. In this placement,  is a rectangle, hence replacing it with its mirror image gives us many distinct solutions.

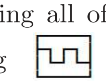
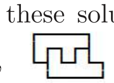
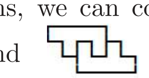
We summarize our results in the following theorem. Among them, we found new types of solutions that cannot be explained in the way of previously known results for $k = 8, 15, 16, 17$.

Theorem 5 *For a rep-tile of the F-shape 6-omino of rep-k^2, we have the following:*
(0) There exists no rep-tile of rep-k^2 for $k = 2, 3, 4, 5, 6, 7, 10, 11, 14$.
(1) Case $k = 8$: All solutions can be obtained by the following way: We first dissect the 384-omino P in one of the ways shown in Figure 7, and then replace each rectangle by  *or its mirror image.*
(2) Case $k = 9$: All solutions can be obtained by the following way: We first dissect the 486-omino P in one of the ways shown in Figure 8 and Figure 9 and then replace each rectangle by  *or its mirror image.*
(3) Case $k = 12, 13, 19, 20, 21, 23, 24, 25$: There exist rep-tiles of rep-k^2. The number of solutions in the case $k = 12, 13$ can be found in Table 1.
(4) Case $k = 15, 16, 17$: There exist rep-tiles of rep-k^2 that include the pattern given in Figure 10.

Proof (0), (3) We can determine the (non)existence of rep-tiles up to $k = 25$ by SAT-based solvers. By using GPMC, we can count the number of solutions (in the existence case) for each k up to 13.

(1), (2) By using NaPS and GPMC, we obtain that the numbers of solutions for $k = 8$ and $k = 9$ are 1358954496 and 51539607552, respectively. We then enumerate all non-concave polyominoes that can be obtained by combining two or three copies of the F-shape 6-omino, and find all tilings using them. After that, we count the number of ways of tilings that can be obtained by filling each rectangle of size 3×4 by  or its mirror image. The numbers of tilings should be at most 1358954496 and 51539607552 for $k = 8$ and $k = 9$, respectively. In fact, we found that we have already listed all tilings since they are equal in both cases. The patterns of solutions are listed in Figure 7, Figure 8, and Figure 9. We use the all non-concave 18-polyominoes obtained by combining three copies of the F-shape 6-omino, however, in fact, only  and  are required to enumerate all solutions for $k = 8$ and $k = 9$.

Precisely, when $k = 8$, there exist six essentially different dissections. When we consider replacing each rectangle by  or its mirror image, we obtain the number of solutions given by Figure 7 is equal to $2^{24} + 2 \times 2^{26} + 2^{27} + 2 \times 2^{29} = 1358954496$ that coincident with the number of solutions obtained by running NaPS and GPMC. When $k = 9$, we have fourteen essentially different dissections. By considering the numbers of rectangles in these dissections, the total number of solutions given by Figure 8 and Figure 9 is $8 \times 2^{30} + 2 \times 2^{32} + 4 \times 2^{33} = 51539607552$ that contains all solutions obtained by NaPS and GPMC.

Checking all of these solutions, we can confirm that we can construct any rep-tile for $k = 9$ by combining , , and  and add one copy of the F-shape 6-omino if necessary.

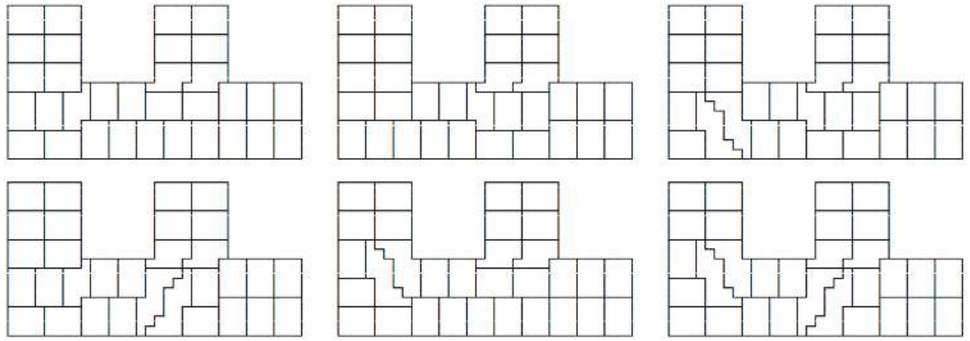
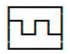


Figure 7: All solutions of rep-tiles of rep-8^2 for the F-shape 6-omino (we can obtain many variants when we fill each rectangle  or its mirror image)

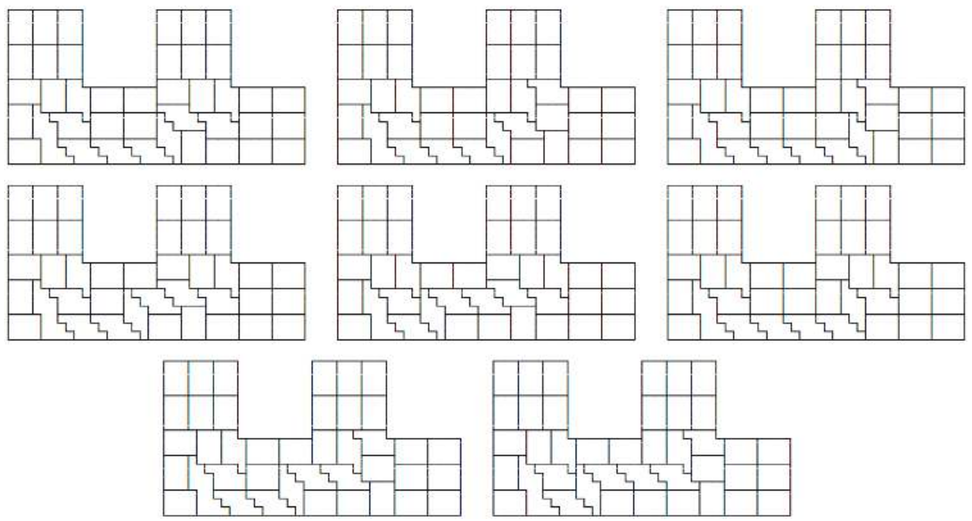
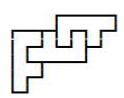
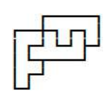
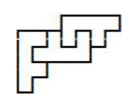
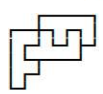
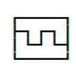
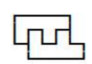
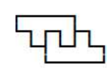
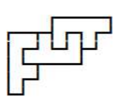
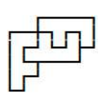
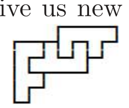
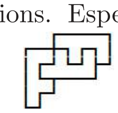
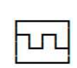
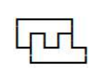


Figure 8: All solutions of rep-tiles of rep-9^2 for the F-shape 6-omino (1/2)

Moreover, the last one copy is added to form  or . Concretely,  is used in the eight patterns in Figure 8, and  is used in the six patterns in Figure 9. That is, when $k = 9$, we can construct any solution by tiling some copies of , , and  with one copy of  or . In other words, these solutions can be represented in the same way of the previously known results.

However, when $k = 8$, we cannot construct all solutions in the way of the previously known results. More precisely, the first two patterns among six patterns in Figure 7 can be represented in this way, however, the next three patterns require to add two copies of the F-shape 6-omino. Moreover, the last pattern requires to add four copies of the F-shape 6-omino. That is, among six patterns in Figure 7, there are only two patterns that can be represented in the way of the previously known results and the other four patterns give us new solutions. Especially, in the last two patterns in Figure 7, we have to place both copies of  and  after placements of copies of , , and

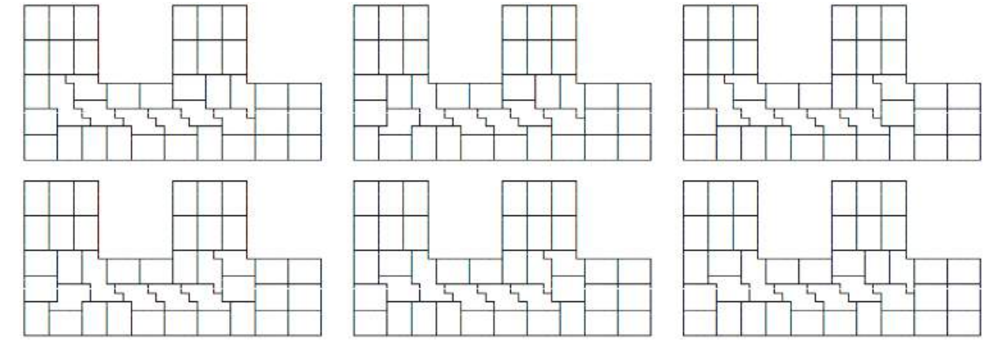
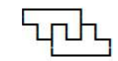


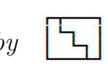
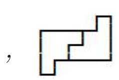
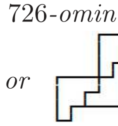
Figure 9: All solutions of rep-tiles of rep-9^2 for the F-shape 6-omino (2/2)

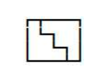
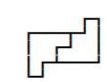
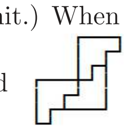
. ∎

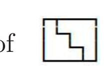
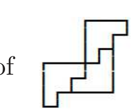
(4) In the case of $k = 8$ or $k = 9$, we can construct all rep-tiles by tiling non-concave polyominoes obtained by combining two or three copies of the F-shape 6-omino. Then, is this common in all the rep-tiles by the F-shape 6-omino? It is not the case. We first note that there exist patterns that require four or more copies of the F-shape 6-omino. A concrete example is given in Figure 10. (There are no rep-tile containing such a pattern in the previously known results.) We searched rep-tiles that require copies of the pattern in Figure 10 with non-concave polyominoes obtained by combining two or three copies of the F-shape 6-omino. Then there are some solutions (Figure 11) containing the pattern in Figure 10 for $k = 15, 16, 17$. They are completely different rep-tiles from the previously known solutions.

4.3 Stair-shape 6-omino

Since the stair-shape 6-omino is not concave (in our definition) contrast with the J-shape and F-shape 6-ominoes, it is difficult to search its rep-tile pattern systematically. However, by generating the unit patterns obtained by combining a few copies of the stair-shape to cancel the zig-zag part of it and tiling the copies of these unit patterns, we succeeded to generate all patterns of solutions for $k = 11$. The results can be summarized as follows:

Theorem 6 *For a rep-tile of the stair-shape 6-omino of rep-k^2, we have the following:*
(0) There exists no rep-tile of rep-k^2 for $k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18$.
(1) Case $k = 11$: All solutions can be obtained by the following way: We first dissect the 726-omino P into one of three patterns in Figure 12. Then replace each polygon by , , *or*  *(or their mirror images). We note that the previously known results are included in Figure 12(a), and the patterns in Figure 12(b)(c) are new solutions that we found in this research.*
(2) Case $k = 12, 13, 23, 24, 25$: There exist rep-tiles of rep-k^2. The number of solutions in the case $k = 12, 13$ can be found in Table 1.

Proof We omit all the cases except $k = 11$ since they were obtained by NaPS and GPMC. (Here we note that $k = 16$ is an exception: the solution in this case could not be obtained by the time limit, however, we could obtain it when we extend the time limit.) When $k = 11$, we perform the search by using three 12-ominoes obtained by , , and . We have three groups by the search.

The first pattern is given in Figure 12(a): It uses 59 copies of  and one copy of .

There are three ways of tiling the left upper green rectangle by 11 rectangles of size 3×4, and six ways of tiling the blue polygon by 23 rectangles of size 3×4. (For the latter blue polygon, there are three ways of

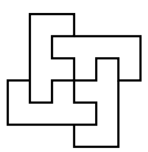


Figure 10: A non-concave 24-omino that requires four copies of the F-shape 6-omino (any removal of one or two copies makes concave)

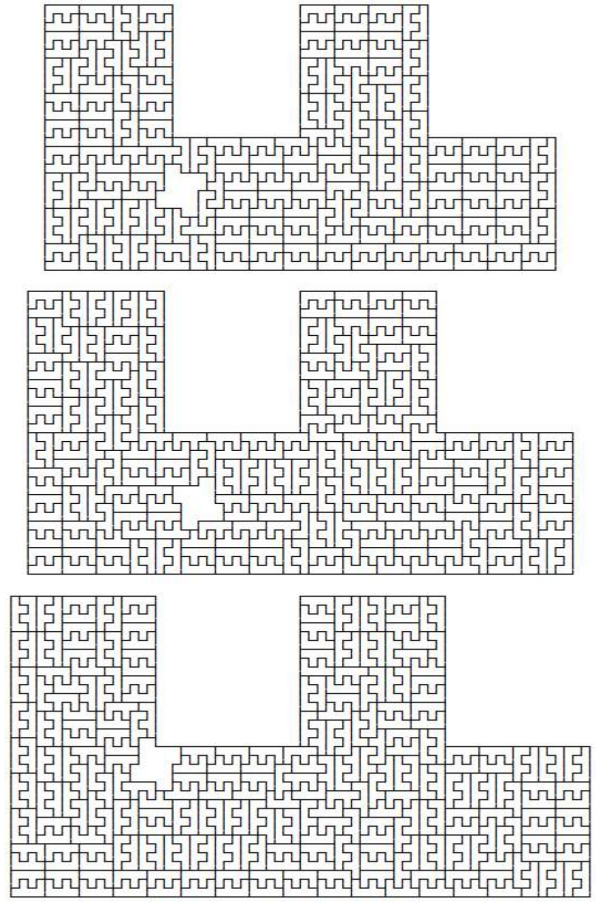
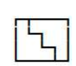
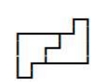
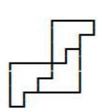
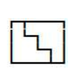
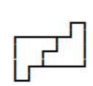
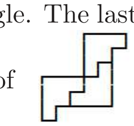
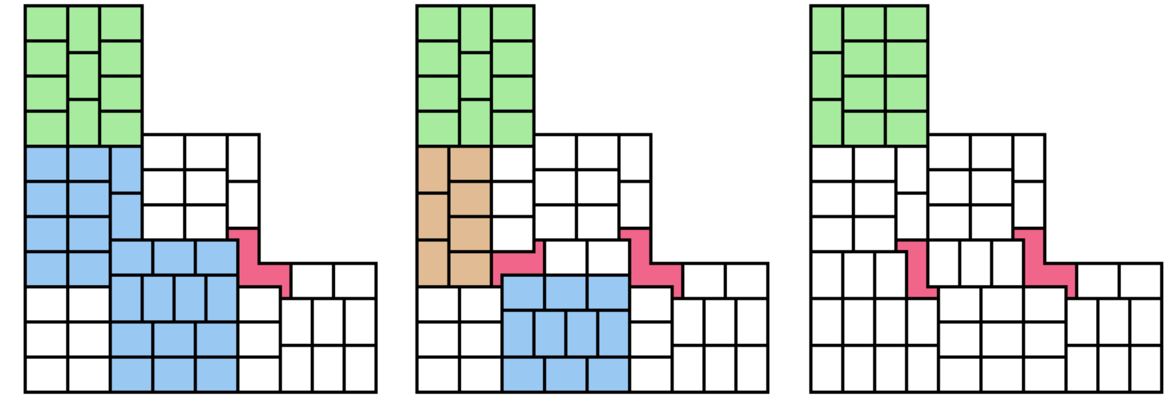


Figure 11: Examples of rep-tiles that contain the pattern in Figure 10 for $k = 15, 16, 17$

tiling of the left upper blue rectangle of size 11×12, four ways of tiling of the right lower blue rectangle of size 12×13, and one in common, which implies six ways in total.) Since we can make a mirror image with respect to the line of 45 degrees, the total number of solutions in the pattern in Figure 12(a) is $2 \times 18 \times 2^{59} = 20752587082923245568$.

The next pattern is given in Figure 12(b), which uses 58 copies of  , one copy of  , and one copy of  . In this case, there are three ways to tile the left upper green rectangle, two ways to tile the central brown rectangle, and three ways to tile the lower blue rectangle. The last pattern in Figure 12(c) also uses 58 copies of  , one copy of  , and one copy of  . It has three ways to tile the green rectangle. In total, the number of solutions in patterns in Figure 12(b) and Figure 12(c) is $2 \times (18 + 3) \times 2^{58} = 12105675798371893248$.

Therefore, when we add all solutions in the patterns in Figure 12(a)(b)(c), it makes $42 \times 2^{58} + 36 \times 2^{59} =$



(a) (b) (c)

Figure 12: Three solutions of the stair-shape 6-omino for $k = 11$

32858262881295138816, which is equal to the number of solutions in Table 1. Therefore, we cover all rep-tiles for $k = 11$. ∎

Acknowledgments

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References

[1] Solomon W. Golomb. *Polyominoes: the Fascinating New Recreation in Mathematics*, Charles Scribner's Sons, 1965.

[2] Martin Gardner. *Hexaflexagons, Probability Paradoxes, and the Tower of Hanoi: Martin Gardner's First Book of Mathematical Puzzles and Games*, Cambridge University Press, 2008.

[3] Martin Gardner. *Knots and Borromean Rings, Rep-Tiles, and Eight Queens: Martin Gardner's Unexpected Hanging*, Cambridge University Press, 2014.

[4] Andrew L. Clarke. Polyomino Reptiles. `http://www.recmath.org/PolyPages/PolyPages/index.htm?Rep06.htm`, accessed in August, 2021.

[5] Donald E. Knuth. *The Art of Computer Programming: Dancing Liniks*, Volume 4, pre-fascicle 5c, September, 2019.

[6] Donald E. Knuth. *The Art of Computer Programming: Satisfiability*, Volume 4, Fascicle 6, 2015.

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Ms Goncalves and the Breakout Rooms

Peter Winkler*

The National Museum of Mathematics (in New York City, at the north end of Madison Square Park in Manhattan) has, for two years and ongoing, been running an email puzzle service called "Mindbenders for the Quarantined." Tom Tsao, one of the fifteen thousand or so subscribers, suggested that I, as puzzle supplier, compose a puzzle based on breakout rooms.

The following proved to be quite a popular entry, perhaps one that Martin Gardner would have liked for his famous *Mathematical Games* column. It goes like this.

"Each day Ms Goncalves distributes her twelve fifth-graders into Zoom breakout rooms containing three or four students each. She has devised a schedule in which every pair of students is together in a breakout room exactly once.

How many days does her schedule run?"

As usual for Mindbenders, the puzzle appeared on a Sunday morning; the following Tuesday there was a hint:

"How many pairs of students are there in all? How many are together on a given day?"

and then on Thursday, a bigger hint:

"Show there are only two numbers of days that make the pairs come out right, and one of them is impossible."

Below, we expand a bit on the offered solution, with the object of illustrating how puzzles like this are most effectively tackled.

For the purposes of this puzzle, a "pair" of students is an *unordered* pair; that is, {Alice, Bob} is the same pair as {Bob, Alice}. It's actually easier

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to count *ordered* pairs, since (in this case) there are 12 choices for the first student in the pair, then 11 for the second; thus, $12 \times 11 = 132$ ordered pairs in all. Since this counts every unordered pair (like {Alice, Bob}) twice, we have to divide by 2 to get the number of unordered pairs: 66.

In general, if there are n objects, then the number of unordered pairs you can form from them is $n(n-1)/2$, often expressed by mathematicians as "n choose 2."

The puzzle statement stipulates that each of the 66 pairs of students appears together exactly once. It's natural to ask: how many pairs are "taken care of" in a day?

There are only two ways to size the breakout rooms on a given day. A "Type A" day, say, has three breakout rooms of size 4 each; a "Type B" day has four breakout rooms of size 3 instead.

On a Type A day, each room has 4 students and thus services 4 choose 2 $= (4 \times 3)/2 = 6$ pairs; thus, the three rooms take care of $3 \times 6 = 18$ pairs of students altogether.

On a Type B day, each room services only $(3 \times 2)/2 = 3$ pairs of students, so the four rooms take care of $4 \times 3 = 12$ students altogether.

It follows that if Ms Goncalves' schedule unites each of the 66 pairs of students exactly once, and if it is comprised of a Type A rooms and b Type B rooms, then we must have:

$$a \times 18 + b \times 12 = 66.$$

To figure out what pairs of values are possible for the numbers a and b, it's useful to divide that equation by 6 to get:

$$a \times 3 + b \times 2 = 11.$$

Now it's pretty each to check that the only possibilities are $a = 3$ and $b = 1$, or $a = 1$ and $b = 4$. Since the schedule runs for $a + b$ days, we see that in the first case, it's a 4-day schedule; in the second, a 5-day schedule. Well, which is it, then?

Suppose it's a 4-day schedule, which has three days with breakout rooms of size 4. Say one of those days is Monday, and another Wednesday. Consider one of Wednesday's breakout rooms; say it contains Carla, George, Miguel and Sumit. Since there were only three breakout rooms on Monday, two (at least) of these four students must have been in the same room on Monday. (Technically speaking, this is an application of the *Pigeonhole Principle*, which says that if n pigeons occupy fewer than n holes, then some hole must contain at least two pigeons.) But then that pair was together on both Monday and Wednesday, which is not allowed.

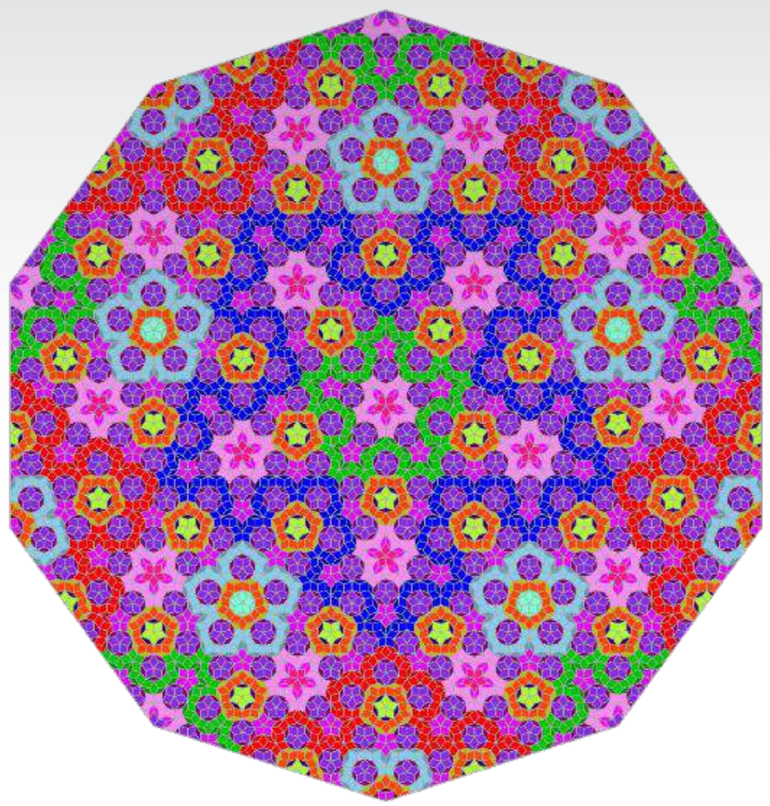
We conclude that no schedule can contain more than one Type A day, and in particular, a four-day schedule (which contains three Type A days) is not possible. Hence, the answer to the puzzle is that Ms Golzalvez' schedule runs for five days.

Wait—if you don't trust the puzzle-poser (and you certainly should not, in this case), you'll want to confirm that there really is a five-day schedule that works. Constructing one is a "hammer and tongs" process, at least for me, but it's not a difficult one. Labelling the students by letters A through L, you can certainly assume without any loss of generality that on the unique Type A day in the schedule, the rooms are $\{A, B, C, D\}$, $\{E, F, G, H\}$, and $\{I, J, K, L\}$. The next day's rooms will need to take one student from each of these groups, so the rooms may as well be $\{A, E, I\}$, $\{B, F, J\}$, $\{C, G, K\}$, and $\{D, H, L\}$. After this some blind alleys are possible, but after a few tries you'll end up with something like this:

Day 1:	ABCD	EFGH	IJKL	
Day 2:	AEI	BFJ	CGK	DHL
Day 3:	AFK	BEL	CHI	DGJ
Day 4:	AGL	BHK	CEJ	DFI
Day 5:	AHJ	BGI	CFL	DEK

Your table may differ—there are many solutions. Next step: let's put an end to the COVID pandemic and break out of breakout rooms!

LEGACY



Tetraflexagon Catalog | Red Deupree | Page 338

Tetraflexagon Catalog / Penrose Pattern Coloring Book

Red Deupree

Abstract

This paper is to formally introduce tetraflexagons that are completely analogous to hexaflexagons as introduced by Martin Gardner in Scientific American over sixty years ago, and place them on the same mathematical platform as their six sided cousins.

In order to be a true flexagon, the thing must show all of its faces without backing up.

This is all based on original work since I discovered true tetraflexagons in 1961. Until recently, I believed mine were original never before published, but that is not so.

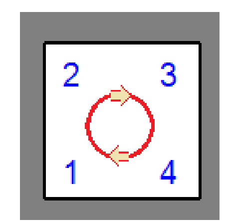
I am proposing a systematic and visual method to distinguish one flexagon from another with the same number of faces, and serve to count distinct varieties.

I am using flexagons to introduce my Penrose pattern coloring system as I can have several generations of the pattern on a single flexagon, or other artistic varieties. In the past, I've written letters on tetraflexagons as that avoids separate pages after front and back and they fold in half to a letter shaped thing.

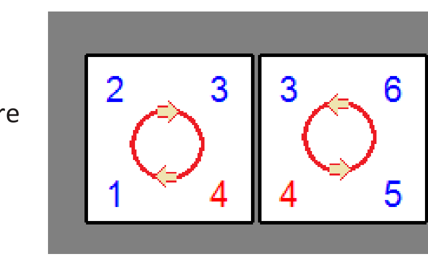
Tetraflexagons

All flexagons can be classified by the sequence in which they show their faces. Tetraflexagons have four sides and show their faces in cycles of four.

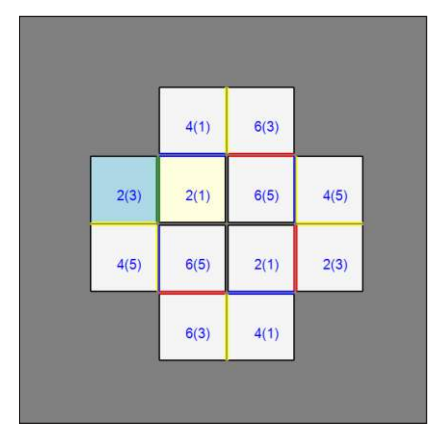
These two flex cycle diagrams are for the primordial pair of tetraflexagons.

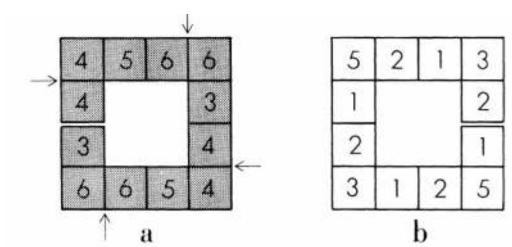


I call these drawings "Signatures". The direction of the cycles is indicated by the arrows and the red numbers are the faces where you have a choice to stay on the current cycle, or switch to the adjacent one.



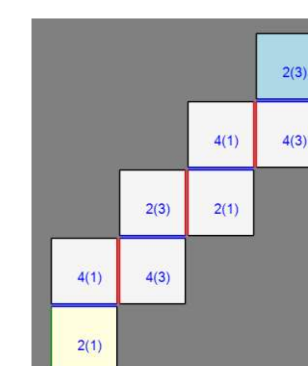
In 1961 I read of tetraflexagons, but I was not able to correctly fold the example from the book*. With scissors and tape I was able make it work like a hexaflexagon. The paper design looks like the pattern on the left.





Hexa-Tetraflexagon map. "Square of squares"

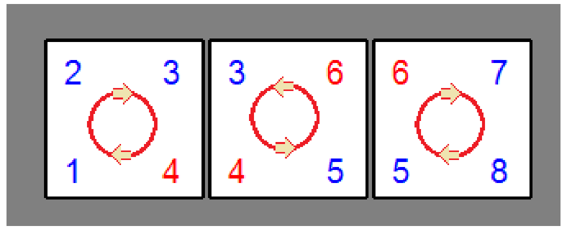
This map is the correct shape for the hexatetraflexagon. The primordial tetratetraflexagon is folded from a shape like one on the right. I found this at the same time.



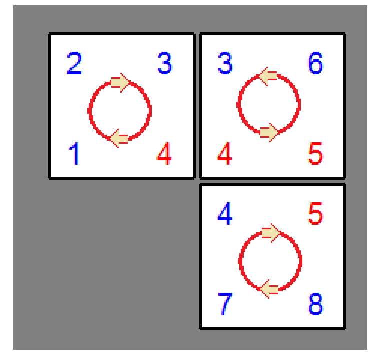
It did not take long to see how to add additional faces to working flexagons after "fixing" the square of squares.

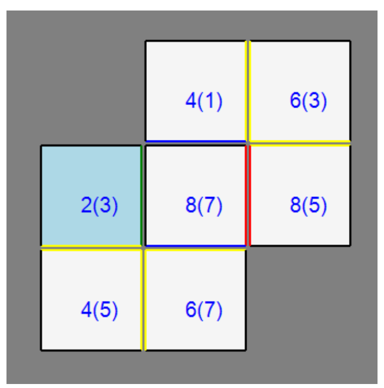
It turns out there are two distinct varieties of octatetraflexagons. I number the faces in the order in which they are added to the signature, but keep all the even numbers on the front and odd numbers on the back. New faces are always added in pairs as no true tetraflexagon can have an odd number of faces.

**By adapting the Internet method for the flat square of squares to the version with the cut and altering the folding instructions slightly, it worked without the glitch from the seamless square of squares. This sort of changes everything.*

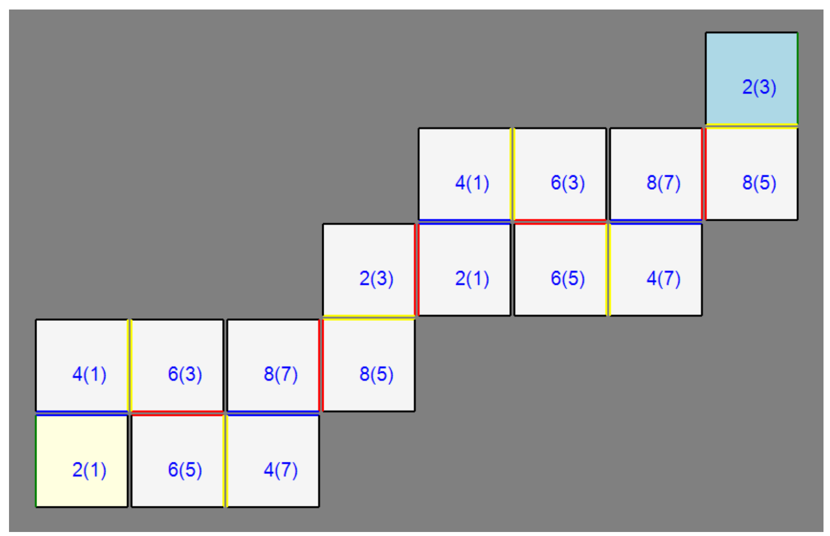


The signature on the left can traverse all eight faces in sequence, whereas the one on the right cannot. They are made from very distinctly different maps. In Martin Gardner's book, flexagons that are able to show all of their faces in sequence were called "street" flexagons, and their signatures are also always straight. Thus there is always exactly one flexagon with a given number of faces that is capable of exhibiting this characteristic.



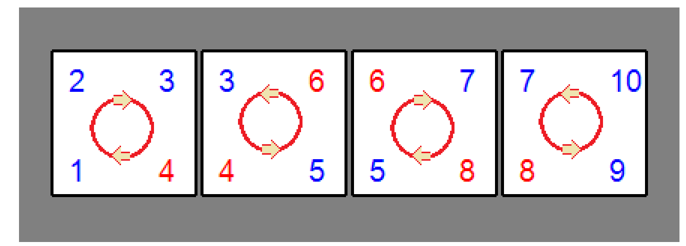


The map for the "street" octatetraflexagon also shares a characteristic with the seven faced street hexaflexagon in Martin Gardner's book: they both have "spiral" maps which when laid flat, have multiple layers. The above map has four layers.

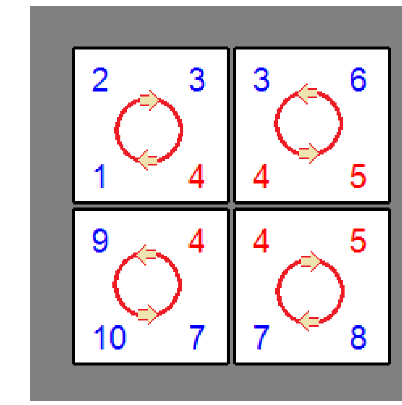


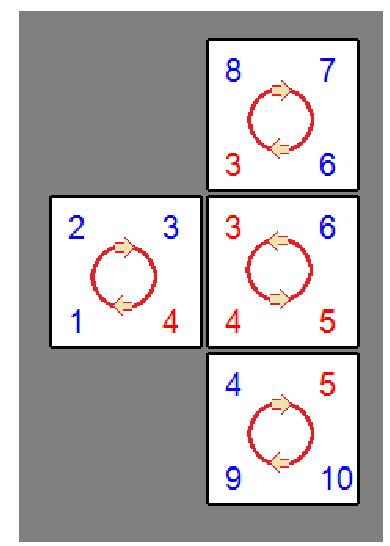
In these maps, I also show the joints between the squares – a blue joint is fold-back from the center and red is fold up from the center, and yellow is fold both ways and be able to have other squares sit between them when folded. Green is for the "glue" joint that has to be made after the flexagon is folded and it will meet with the glue joint at the other end of the strip. By folding the squares together in reverse face order, two at a time, the partially folded strip will resemble the strip of the flexagon from which it was made. This is also true for hexaflexagons. The numbers in ()s are the face numbers on the back.

Carrying on from octatetraflexagons, it turns out there are five distinct varieties of decatetraflexagons.

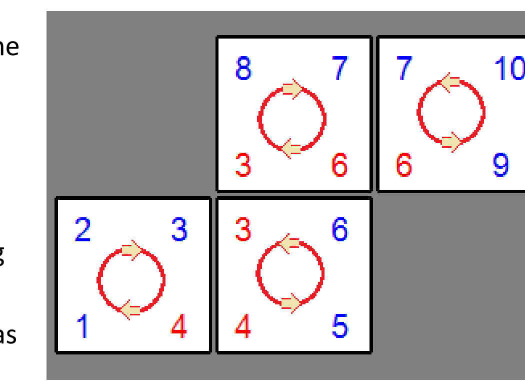


Their signatures are familiar to any of us who ever played Tetris.

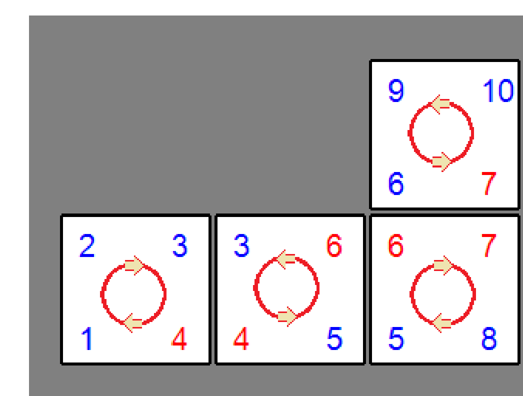




This is the "aha" moment that led me to come up with connection between tetraflexagons and polyominos almost sixty years ago after reading Martin Gardner's original edition of 'Second Scientific American Book Mathematical Puzzles and Diversions'. Being able to visually enumerate the varieties and seeing this apply to hexaflexagons as well was the breakthrough to my method for explaining the number of varieties of flexagons with a given number of faces. I would like to know if this line of thought is mathematically valid, but it does seem to work and is easier to visualize.

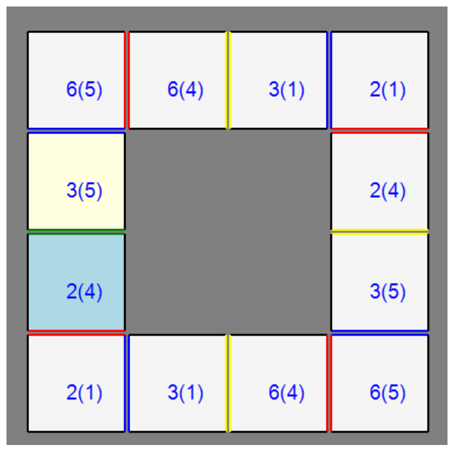


The square signature above is where the first difference shows up as it is actually a U shape and thus there is a way to add a square on top of another in the signature by adding a pair of faces on the 9-4 edge, which does work, thus making for sixteen varieties of dodecatetraflexagons instead of 12 for the number of pentominos (the U, being asymmetric, has four distinct ways to add another square, three of which lead to the "P" pentomino but have distinct flexagons to match. The maps for the five of these, and a few higher order flexagons are in the tetraflexagon catalog.



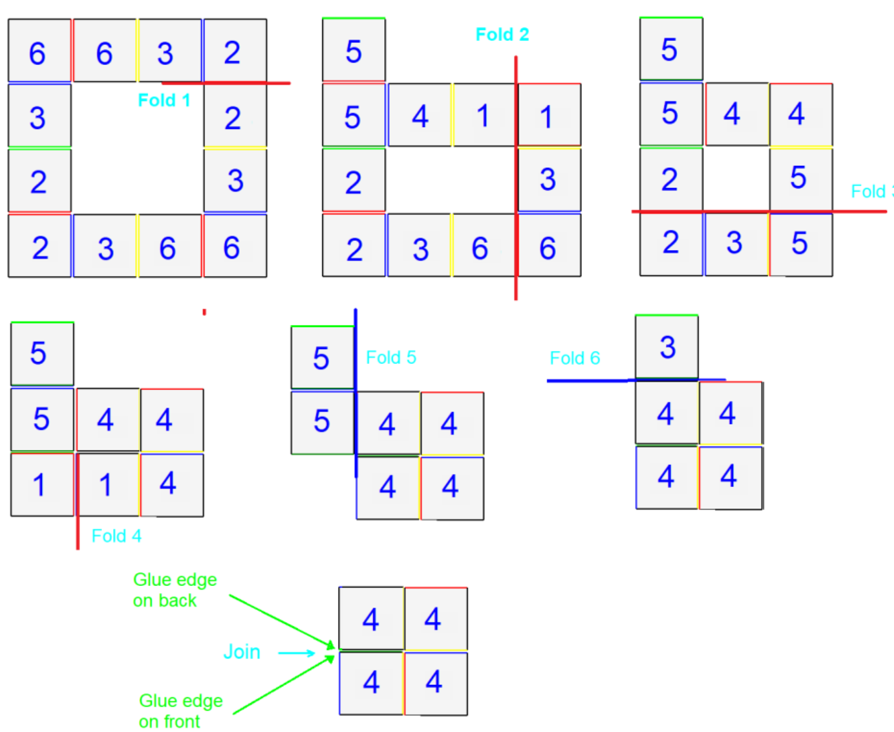
I claim that this process can produce all possible true tetraflexagons.
Maybe? I think so, still.

Right and Left handed forms of flexagons, maps, and signatures are possible. I claim that if the signature is the same, the map will be the same, regardless of the order in which faces get added.
Because the primordial tetraflexagon does not have three squares in a row in its map, and the process for generation tetraflexagons with more faces never produces three squares in a row in its maps.



So what to do with the square of squares (and its relatives)? It works, but it is sort of wonky and has a flex cycle where it half opens and has a configuration with multiple squares on both sides and won't open. If joint is pulled on, the whole thing comes apart and turns back into the square of squares. Flat. But it does work, and its flex cycle is the same as my hexatetraflexagon that is not flat, but has two or more twists in its chain of squares.

I adapted the method for folding the seamless square of squares to Martin Gardner's square of squares and it produced a working hexatetraflexagon identical in its signature from the one I discovered in 1961.



This is what it should look like after each fold as the flexagon is assembled. Note that folds 2 and 3 are both joints at once.

The joints in the map follow the same principle, red is one-way up, blue is one way down, and yellow is two way.
In this map, adapted from Martin Gardner's map, the split is on the two-way joint on the left. In these folding instructions, I am showing only the face up number you should see if the faces are numbered. I recommend numbering the faces with Post-it® notes.

Making Flexagons

The process for making flexagons is fairly straightforward. Fold the faces together in reverse order, two at a time, and make sure that the partially folded map looks like one of the maps with fewer faces.

The final primordial tetraflexagon is folded by folding the 1s and 3s together so 2s are on the front and 4s on the back. Fold the 2s together, where they meet needs to be joined with a one-way joint facing in. The easiest way to do this is carefully open the flexagon to the 3 face, turn a quarter turn to the 2 face which will now have its two halves neatly divided and open to be joined. This last joint can never be automatically perfect, but if you can align the last edge for the one way joint, the whole flexagon will be the best it can be.

All of my maps will end up with this step. (The square of squares does not)

A good flexagon should be able to be flexed through all of its faces repeatedly without wearing out, coming undone because of pulling the difficult folded stacks past each other in certain cycles, usually the 1-2-3-4 cycle because the two-way joints next to the "4(1)" faces, present in all my maps, has to fold all the other folded faces at some point to complete the 1-2-3-4 cycle.

Mathematical flexagons have zero thickness material that is inflexible and joints that are perfectly flexible, thus they fold together neatly with no need for engineering, real flexagons with more than 8 faces need more serious engineering.

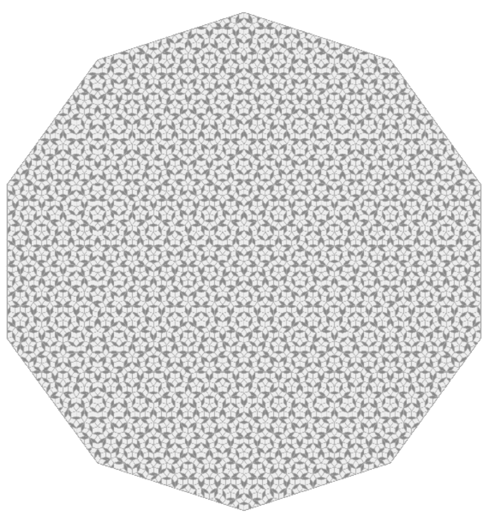
I have made nice "show" flexagons with four inch squares (and triangles), but I chose 2 inch squares and maps that could be printed on a single sheet of paper as the process for making skinned flexagons with cardboard insides and flexible joints is slow, expensive and error prone still. Even with exact printing, cutting and folding errors can ruin a paper flexagon, and any flexagon with more than 6 faces is going to have "difficult" cycles where it has to be carefully coaxed to open to make sure nothing tears or breaks.

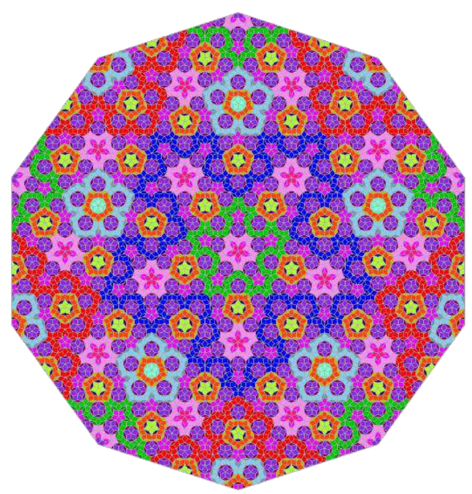
Penrose Patterns

I have chosen to use my Penrose Patterns for decorations

I have written my own app that generates these patterns so I could implement my coloring system, which takes note of the fact that Penrose patterns form rings of shapes (kites, darts, skinny and fat rhombuses). My app counts the number of shapes in each ring and then assigns a color to each distinct count and colors rings of that could with the color.

Like when a black light shining on certain rocks, reveals patterns, so when the color method is applied to a random patch of the pattern, its structure is revealed and is the basis for all of my Penrose art.





The two color patterns come out of the box for free as skinny and fat is a property of every triangle.

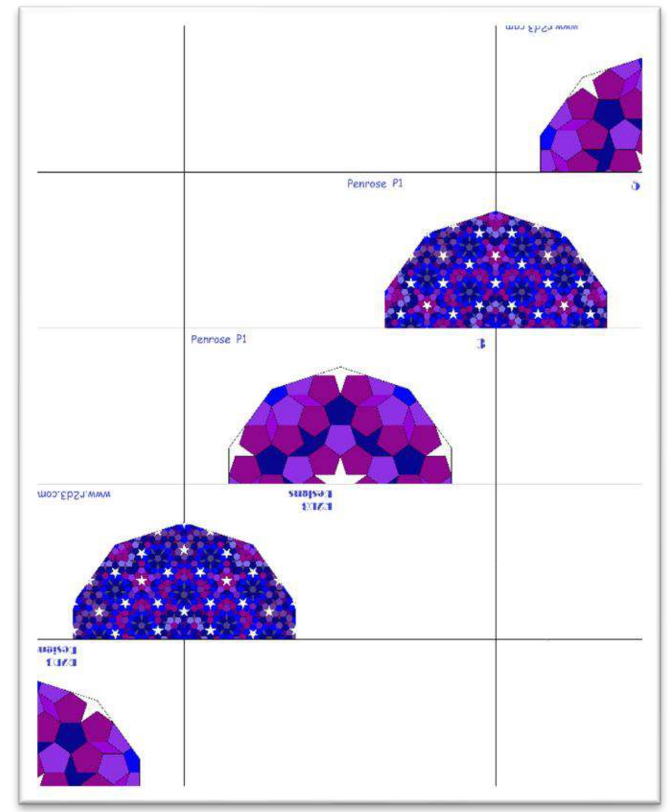
My app assigns a distinct ID to each triangle, so it can find adjacent triangles (halves of the familiar rhombuses, darts or kites) and has a database of distinct vertices, including the center points on the triangles so it can count the rings.

The algorithm for dividing triangles to make the Penrose patterns is described in Wikipedia under “Penrose Tiles”

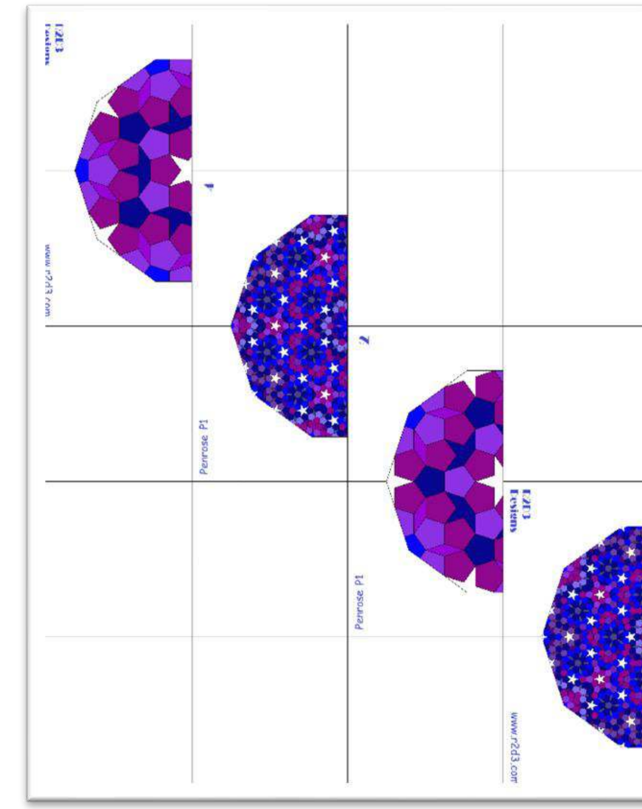
Having my own program has allowed me to bend the rules, make spheres, as well as all kinds of artworks.

Decorating Flexagons with Images

If you want the flexagon to have all four faces form an image like Markus Götz’s folding puzzle, the orientation matters. These are the front and back of one of the flexagons in the gift exchange as they are printed. The flexagon has to be cut and the print has to be two sided and the sides must match up.



Side 1



Side 2

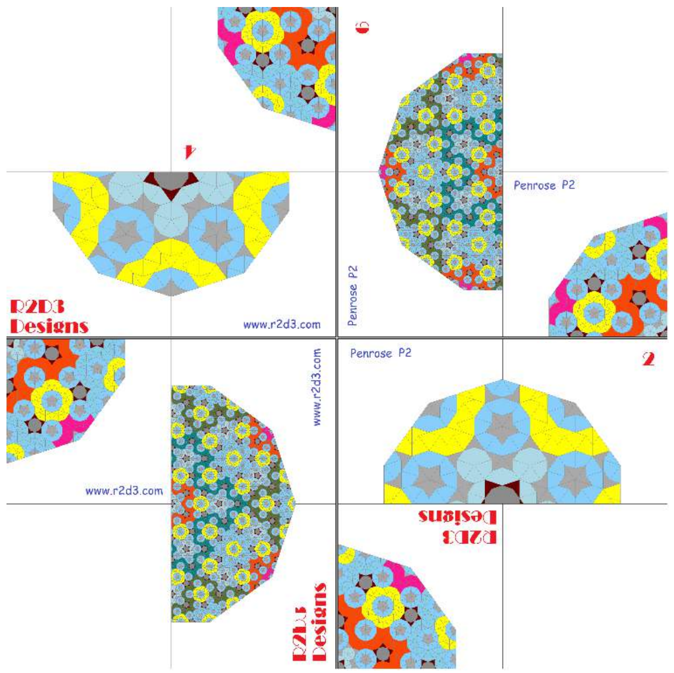
This one shows the P1 Penrose pattern, which is the first one discovered by Sir Roger Penrose.

This is for the tetratetraflexagon, the primordial one of its kind

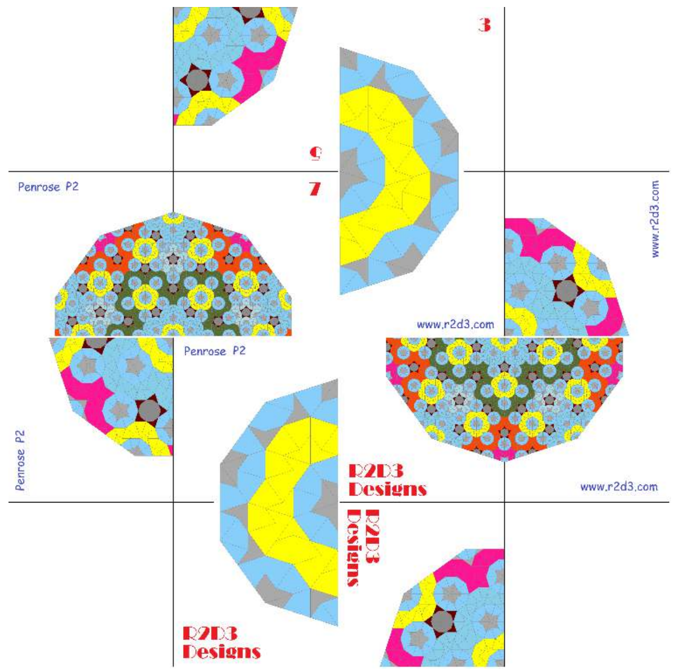
This pattern is designed to be rotated counterclockwise except for the last flex in the cycle when you rotate clockwise (so the images stay right side up.) If the image is not a decagon, you are looking at the “back side”. Turn it over and turn it right side up and it will show its faces in order.

The hexatetraflexagon is the first kind I discovered in 1961

This decoration was chosen to represent the P2 pattern, the one that Martin Gardner used in is chapter on Penrose Tiles.



Side 1



Side 2

In order to make a nice flexagon with more than four faces, the "outside" edges that are on the "inside" of the pattern need to be trimmed a bit, and a bit of extra spacing for the two way joints.

I recommend creasing the two way joints of a paper flexagon well before final folding. As you fold the faces in pairs in reverse order, make sure the image aligns properly and your flexagon will behave reasonably. Newly made flexagons need to be "broken in" by flexing them through all their cycles and making sure that the stacks set so they don't bind and press the folded flexagon and the irregularities will work themselves out (within reason)

Errors are additive and even if the flexagon is perfectly printed and cut, small folding errors can lead to edges poking out or the images not merging. If the two sides are not exactly matched, the odd or even faces will always be a bit off as all the even faces are on one side and the odd faces on the other.

Are You a Mathematician?

Dana Richards

I first met Asimov on the sidewalk and introduced myself.
He asked "Are you a mathematician?"
"No, I never took any courses in math."
He said "You mean you are working the same racket as I am?"

While it is not true that Martin Gardner never took any course in mathematics (there is one on his University of Chicago transcript) the question remains: was Gardner a mathematician?
We approach this question from several directions.

Gardner never claimed to be a mathematician and always referred to himself as a "reporter." His non-fiction writing reported on what others had discovered. He often said that it is precisely because he struggled with mathematics that he was able explain it so well to others. He knew where the pitfalls were. But with his struggles he got his hands dirty and made some discoveries on the way.

In high school he wanted to be a physicist and was strong in math. Also, he had a phenomenal collection of mechanical and mathematical puzzles.

8 HOBBIES SEPTEMBER, 1934

A PUZZLING COLLECTION By MARTIN GARDNER

AS a youngster of grade school age I used to collect everything from butterflies and house keys, to match boxes and postage stamps—but when I grew older, and began to realize that it would probably be some time before I ever achieved a complete or even an outstanding collection (particularly with my initiative and financial resources). I began to specialize in various fields—only stamps bearing the picture of a map, or only match boxes containing black matches

lished in 1893 and now hard to secure. Most of the puzzles described are no longer being manufactured. Ottokar Fisher's German work, "Illustrated Magic," recently translated by the present editor of "Liberty Magazine," devotes its entire last chapter to mechanical puzzles. Dudeny, the English puzzle genius, interspersed descriptions of many mechanical puzzles throughout his works, although the laté Sam Loyd of America seemed to have little interest along this line.

the many knots in a cord is actually the joining spot of two separate cords. Untying the knot effects the solution, and the rest of the wooden parts are just there to confuse you. Another puzzle must be soaked in water before it can be worked, and still another requires a special instrument for its solution, an instrument which you do not give to the victim!

If one practices he can acquire some astonishing proficiencies. A small lad once picked up my spider web

We can find in his magic writings (nearly every year from 1930 to 2010) an early acquaintance with the popular math books. In fact, when he moved to New York after the war he attended a lecture series sponsored by Yeshiva University led by Jekuthiel Ginsburg. Ginsburg was a strong proponent of popularizing mathematics. The time period was 1948 to 1952. He was seeking ideas for magic and satisfying his interest in math. It was for this audience he wrote a series of four articles for *Scripta Mathematics*, which were later expanded into his *Mathematics, Magic and Mystery* (1956).

One way he showed that he working through an issue rather than just reporting on it, is his proposing open problems in mathematics journals. He did this for decades, starting as early as 1947. An example is:

Dissection of a Regular Pentagram into a Square

E 1309 [1958, 205]. ***Proposed by Martin Gardner, New York, N. Y.***

Dissect a regular pentagram (five-pointed star) into no more than nine pieces which can be reassembled to form a square. Pieces may be turned over.

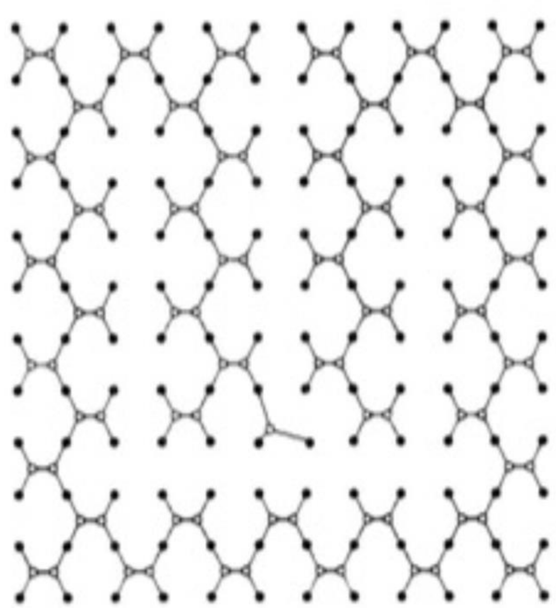
Occasionally he would offer cash prizes,

His sole-authored math articles were fewer in number and reported on his experimentation.

TILING THE BENT TROMINO
WITH N CONGRUENT SHAPES

MARTIN GARDNER

However, he did coauthor several articles with well-known mathematicians. His contributions were often suggestions and examples that kept the work going forward with the occasional theorem. It should be remembered that Gardner was almost never a coauthor. In these cases he was just happy to contribute for the experience of working with his creative friends.



Steiner Trees on a Checkerboard

FAN CHUNG
Bellcore
Morristown, NJ 07960

MARTIN GARDNER
110 Glenbrook Drive
Hendersonville, NC 28739

RON GRAHAM
AT&T Bell Laboratories
Murray Hill, NJ 07974

$47x + t = 130.33824\ldots$

Sylanus Thompson text on calculus was reissued my Gardner, *Calculus Made Easy*. The text used an older approach based on infinitesimals rather than limits. Rather than just adding an introduction Gardner added several chapters explaining the background for the modern reader. In addition to all the above Gardner also wrote about the Philosophy of Mathematics, starting in 1950 with “Mathematics and the Folkways” (*Journal of Philosophy*).

The prevalent view is that Gardner was no mathematician and that when he started with *Scientific American* in 1957 it was all new to him. In fact, he was always interested in math, but not as a job. He enjoyed it. He played with it and studied it. Obviously, his early columns were less sophisticated and his grasp of issues blossomed over the years. As this happened his circle of correspondents expanded. To keep up his end of the discussions his mathematical knowledge also expanded,

If a “mathematician” is someone who derives theorems he was not one. However, very few theorems are fundamental, they instead advance the field in a incremental fashion. Gardner, on the other hand, acquired a broad knowledge and used that to propel mathematics forward, influencing countless people to enter the field. If a “mathematician” is some who contributes to the success of mathematics, then Martin Gardner was one.

Dear Mister Gardner: Apocryphal Letters from Children to Martin Gardner

Rodi Steinig, M.S.Ed.

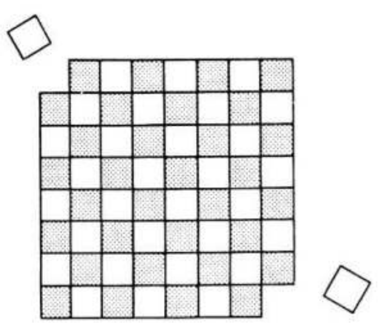
ABSTRACT: In the fall of 2014, a group of six students got together for an hour per week to celebrate the Gardner Centennial. The students were all ten years old. They met in a Math Circle to explore the Gardner's life, influence, and mathematics. The mathematical goal of the course was to develop students' mathematical thinking by seeking patterns when none are obvious and by seeking ways to crush seemingly-obvious patterns that aren't really patterns at all. The students experienced great joy while doing so. The facilitator kept a written record of the students' reactions, work, comments, and questions, all of which have been reorganized by the facilitator into a series of letters to Martin Gardner. The letters are paraphrased with some direct quotes interspersed, with the hope that Mister Gardner would have enjoyed reading them.

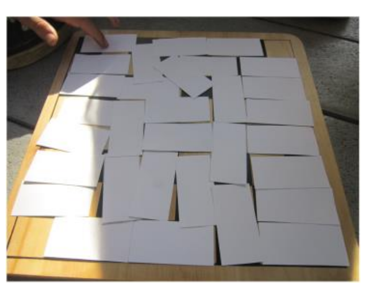
September 23, 2014, Week 1

Dear Mister Gardner,

We're a group of six kids meeting six times in a Math Circle in Philadelphia. We're ten and eleven years old. We're celebrating the Martin Gardner Centennial.

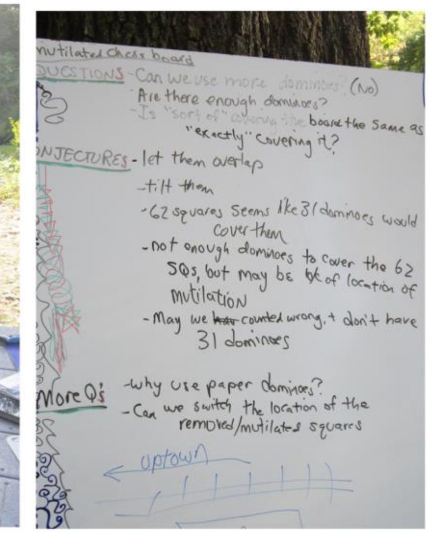
Our facilitator Rodi said that you mutilated a chessboard. Strange. In your problem "The Mutilated Chessboard," you asked if we remove opposite corner squares, can we cover it with 31 dominoes?

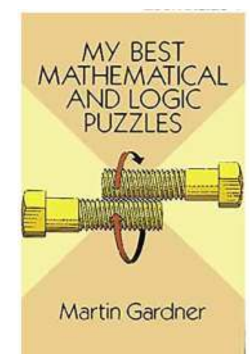


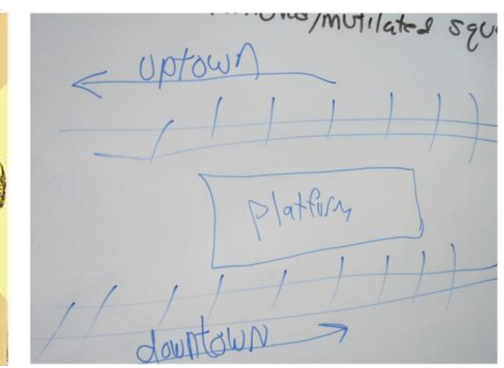


We had a lot of questions about it. First of all, can we please use more than 31 dominoes? We're wondering whether 31 dominoes is enough to cover the whole board? Why are we using paper, instead of real dominoes? And can we switch the position of the removed squares?

We had a lot of ideas about how to solve this problem. We could let the dominoes go on top of each other. We could tilt them. We do think that 31 dominoes seems like enough to cover 62 squares since 31+31=62, so it should work. But, it might not be working b/c there are not really 31 dominoes here (let's count them!). It doesn't seem to work – maybe because of where the removed squares are? Wow, Mr. Gardner, you really know how to write a great question. At this point we're thinking that it doesn't work with diagonal corner squares removed, but it does with corners that are next to each other.







Rodi also showed us your problem Bronx versus Brooklyn. She gave it to us with a new wording but kept the math the same. After half an hour, we came up with the same explanation that you did! We figured it out because every time someone said something, even if it was wrong, it helped us with our next idea. We were sad that time was up in our class. We asked Rodi, "Are there more problems like those?"

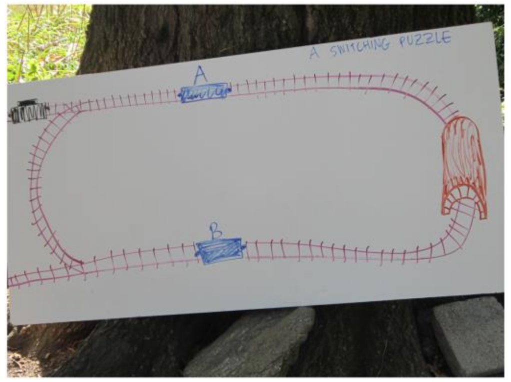
Sincerely,

Crystal, Candace, Akira, Hudson, Mckenna, and Lily

September 30, 2014, Week 2

Dear Mister Gardner,

Rodi put on the whiteboard a drawing of your problem "A Switching Puzzle." We did not let her tell us what the question is; we wanted to guess it.



We asked a lot of questions and came up with a pretty good question for this picture, but it was not the question you asked about. We got to work on your question by tracing train routes with our fingers. Unfortunately, that erased the drawing! So we started to use the markers as trains, which worked well since they can be attached to each other.

All of us were not at the board, though. Lily sketched it out in her notebook and Crystal was just sitting there, seeming to do nothing. Turns out she wasn't doing nothing. At the same time, both Lily and Crystal jumped up and said "I think I've got it!"

"I think I've got it," is the thing we say the most often in Math Circle. The next most-common thing we say in Math Circle is, "Oh wait, never mind." We all did end up at the board working together to come up with the same solution that you did. It was a combination of working together and working alone, like real world work. Every time one of us said something, it made another one of us think of a new idea.

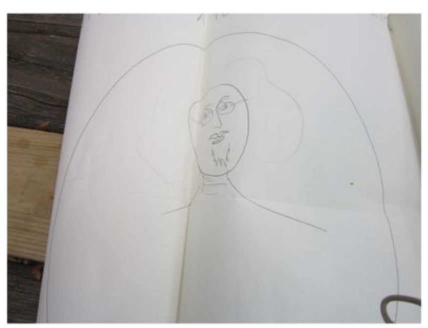
Next, Rodi showed several maps that were like a lot of other paper maps, easy to unfold and hard to refold in the same way. We're used to Google maps, so we didn't know this. After that, she said, "I just can't figure out a certain puzzle no matter what I try." That got us so excited! The puzzle was called The Folded sheet by Henry Ernest Dudeney that you wrote about in one of your books. Rodi said she worked and worked on this puzzle. She read the instructions in the back of the book. She watched a video on YouTube that had hints but not the answer. She was hoping that we could help her.


We tried and tried too. We didn't get very far – just 1 then 2 in their places. Mckenna made her own "map" out of a new paper with larger squares. Soon everyone could get 1, 2, 3. Candace made a new map with tiny squares. After a while some of us could get to 4. Then we all agreed that there must be some kind of weird fold, maybe like a diagonal. Some of us tried this and made it to 6. Hudson even got to 7 but lost 8 in the folds. By then, we were so frustrated! We decided to take our maps home. We made Rodi promise to bring the book with your instructions next time.

Rodi thought then would be a good time to give an "easier" puzzle: your "Folding Money Fun." You show in pictures and words how to take a dollar bill and fold it one way and unfold it in the same way, but George Washington ends up upside down. Rodi asked us to draw Washington's face on the back of our maps, which were the size of a dollar bill. We needed extra time to get Washington's hair just right, but Rodi told us not to worry about this because it didn't have to do with the math we were doing.

The question you ask is "Why does the bill turn around?" It would have been better had Rodi asked us "Can you follow these directions and get the same result?" You say that "If you have followed the illustrations exactly…" But it isn't so easy to follow the instructions exactly. It's hard to "fold forward and to the left." Especially 'cause we're only 10 and we recently learned our directions.

Only two of us were able to follow the directions. The rest of us got so, so frustrated! The two of us who could do it were sitting closest to the book. Rodi should have given each of us our own set of instructions! But she wants us to do everything collaboratively, with everyone doing the same thing at the same time, and asking each other for help. If we have to do things that way, maybe this just isn't a good problem for our group. So far, "Bronx vs. Brooklyn" seems the be the most successful puzzle for our group – we were super-interested, thinking hard, working with our friends, and nothing that was hard for us to do with our hands like folding or cutting. "A Switching Problem" was a great problem for us too – since we talked more than using our hands. Next time, Rodi plans to make one very large dollar bill for everyone to fold together. But then she would get to have all the fun drawing George's hair!



Anyway, the two of us who did figure out how to turn George upside down did realize the answer to your question. The rest of us struggled with the folding. Crystal got so frustrated that she doesn't want to do paper folding ever again. The rest of us are willing to give it a chance. And we are not blaming you. At the end of today's class, we finished with some of your riddles and word games, so we all left with a smile.

Sincerely,

Crystal, Candace, Akira, Hudson, Mckenna, and Lily

October 7, 2014, Week 3

Dear Mister Gardner,

It turns out that the last two weeks Rodi was telling a few of your problems from what she remembered, instead of reading them directly from your books. So today, we got to hear your versions of some of the problems. We thought the way you talked was confusing! (But Rodi thought that your wording was incredibly clear and that our vocabularies are lacking. We are only 10, after all.) When we heard "A Switching Puzzle" in your words, we couldn't remember how we solved it before. We wanted to make sure we were following all of your rules, so we re-solved it. That was fun!

Rodi asked us to go back to "Folding Money Fun." This time she had prepared a big paper dollar bill. We were very excited, so we fought over who got to draw in George Washington's hair. Each of us did a different section of the bill and folded while listening to the directions. We did it a few times. The good news is that by this time, we were all sure of how to trick our friends (and parents!) with this puzzle, and we also knew why it works. The bad news is that we couldn't all hold the large bill at the same time. Rodi should have made it at least 4 or 5 feet long because in this problem, the way it's facing matters – we have to all be on the same side for the folding to be able to follow along.

Next we went back to "The Folded Sheet" with an extra-large number map. Rodi read aloud your instructions as we folded. Unfortunately, one of us, we're not naming names, got so excited that they took charge of folding the map. Another of us got frustrated with that bossiness and walked away. And another played the peacemaker and brought that student back.

Rodi asked everyone to work together, but it was too late. She likes to try to become invisible in Math Circle, but should have told us what to do when things got too intense. We worked together but we were grumpy. One of us suggested that everyone fold their own map. You know, Mister Gardner, that Rodi does not like doing it this way, that it isn't collaborative enough for her. She didn't have extra paper anyway, so we tried doing one large map for a bit without arguing but also without a solution.

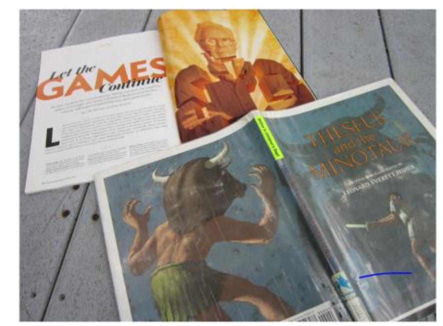
Did we mention that we were having class outside on picnic tables? On one of the tables today was a large, laminated copy of your "Maze of the Minotaur" with some smaller copies of something that looked the same. We moved to that table without being told when we lost interest in the Folded Sheet. Since most of us are big fans of Percy Jackson's adventures, we asked repeatedly "Is this the labyrinth of Theseus?" All she would say is "What do you think?" and then we argued about it.

Most of us started tracing the large path with our fingers. Our fingers walked all over it trying to figure it out. While we were working, we heard that you said *'No one has ever drawn a maze that looks easier to work, but actually is so difficult.'* Rodi thought she was a smarty pants because she initially thought she had solved it in about 2 minutes, so she thought we'd be able to do it without getting annoyed. She drew a larger version for us and she couldn't solve it. Smarty Pants Rodi thought she must have drawn it wrong. But she hadn't. …she started to realize that you were right. … She started asking us the questions she was wondering so that we could help her figure it out…. It turns out that she didn't have to ask us questions at all because we were wondering the same things:

- Is there actually a pattern in the solution?
- What's the difference between a maze and a labyrinth?
- Is the solution manageable?
- Are there useful strategies for drawing the maze?

After a while, finally one of us successfully solved the maze. And guess what: we weren't arguing anymore! We were very calm. Do you think this calmness was from walking our fingers in a labyrinth?

Rodi read aloud some parts of the new Scientific American article about you. We enjoyed hearing about your playfulness. And it was nice, too, for us to see a picture of you. You are not a guy from ancient history!



Rodi ended class with one of your math puzzles that involved no objects – no pen, no paper, no chessboard, no poster, no giant dollar bill, no nothing. Scrambled Box Tops. It went like this:

"Imagine that you have three boxes: one containing two black marbles, one containing two white marbles, and the third, one black marble and one white marble. The boxes were labeled for their contents – BB, WW and BW – but someone has switched the labels so that every box is now incorrectly labeled. You are allowed to take one marble at a time out of any box, without looking inside, and by this process of sampling you are to determine the contents of all three boxes. What is the smallest number of drawings needed to do this?"

We made some progress toward the solution, but then we were out of time. Rodi really needs to teach it differently next time so that everyone is talking; today only 2 of us talked the most. And they didn't even agree. They worked on the problem totally differently. One of us drew it, the other was thinking about it. Both ways actually helped. How did you figure it out, Mr. Gardner?

Sincerely,

Crystal, Candace, Akira, Hudson, Mckenna, and Lily

October 14, 2014, Week 4

Dear Mr. Gardner,

Rodi opened up a box and said "I'd like to show you my creatures,"

"That's a Go board," protested Crystal.

Rodi said "No, this is the habitat of my creatures. It does resemble a Go board, but it's not one. I'd like to tell you about how this species of creatures exists over time." She placed a few black counters on the board and said "They can live if they have 2 or 3 neighbors."

"What if they only have one?" asked Akira.

"Then they die." Said Rodi

"Of loneliness," added Mckenna.

"What if they have 4 neighbors?" asked someone else.

"They die." Rodi went on like this for a few minutes. We asked a bunch of questions and figured out the rules of death and survival. "In real life," Rodi said, "something else happens besides survival versus death."

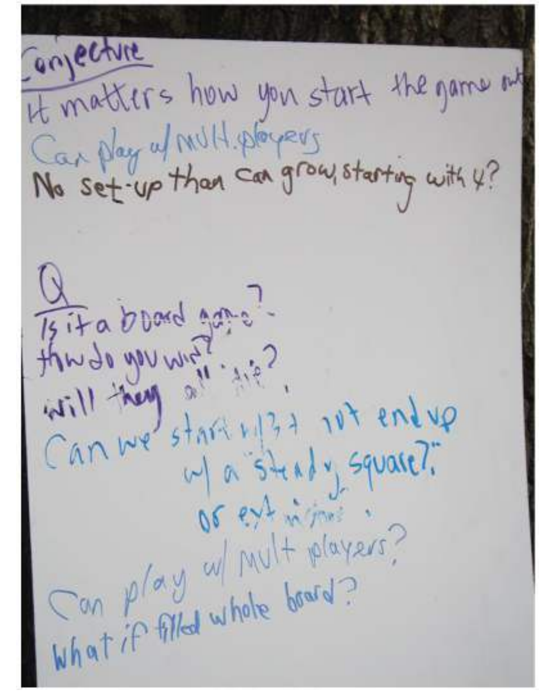
After a couple of guesses, Hudson said "new ones are born!" He was excited. Then Rodi explained the rule about births – they only occur in a cell/location with exactly 3 neighbors.

"Is this something Martin Gardner invented?" asked both Lily and Mckenna.

"No, but… " Rodi started to say, and before she got the sentence out, we all said "Martin Gardner made it famous." We get what you do. Rodi showed us your article about it.

"Are you going to ask us if we can fill the whole board with creatures before they die out?" asked Lily.

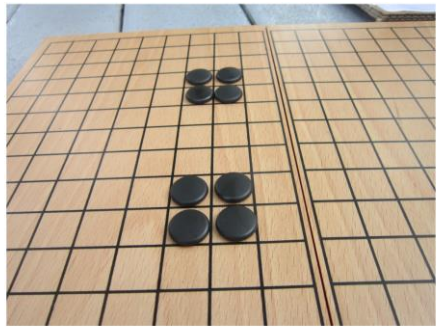
Rodi said "That's a really interesting question to explore. Today, though, I wasn't going to ask you any questions. My plan is just to show you the creatures and the rules they live by, and have you all come up with questions."

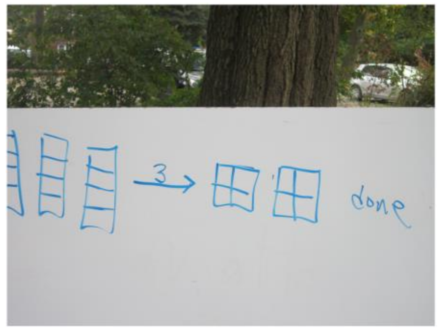


"Is it a board game?" asked Mckenna. Rodi said let's try it out and see what we think is the answer to that. We started to play and had a bunch of questions and ideas:

- "It matters how you start the game"
- "How do you win?"
- "Will they all die?"

We played more and saw that some initial positions resulted in immediate or quick death to all, some seemed to keep going, and some ended up in a stable position. We called that a "Steady Square." We wondered if there are answers other than a steady square or death?





While we worked, Rodi told us about how this game was one of the early ways people used math modeling to predict outcomes for populations of species. “Oh no, Ebola!” we said. “Yeah, people with the wrong number of neighbors might die of Ebola!” we said. We were a bit nervous so asked Rodi “Is that how it works in real life?” She told us that it’s just a game, and its purpose mainly is to get us thinking. It isn’t human lives and deaths. Remember, she said, many factors influence our survival. All this game specifically examines is population density.

Almost an hour later, we were still at it, and then it was time to go home.

Sincerely,

Crystal, Akira, Hudson, Mckenna, and Lily

October 21, 2014, Week 5

Dear Mr. Gardner,

Candace wasn’t in class last week and was early for class today. She walked in and saw our work on the board from last week. She asked, “Why does it say ‘die?’”

Crystal and Lily were also early for class, so Rodi told Candace to “Ask them.” Crystal set the board up for Life. Then Lily explained the rules.

Rodi seemed to think that some of Lily’s explanation didn’t make sense, so she rudely interrupted. “I think Lily is doing a <u>fine</u> job of explaining,” said Crystal. Crystal was right. Rodi did need to butt out. Rodi sometimes has that annoying teacher habit of butting in. She needs to work on that.

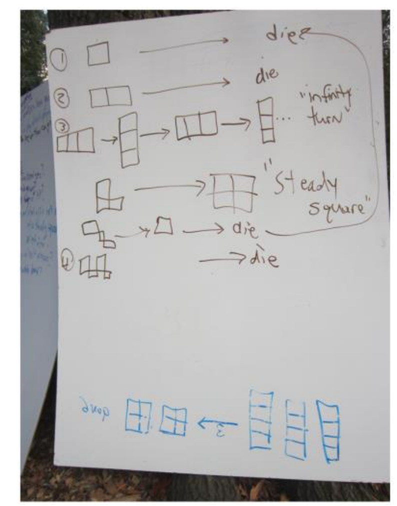
Then Hudson got there and said “I’ve been thinking about it all week and still can’t figure it out.”

“<u>What</u> are we trying to figure out?” asked Candace.

Before anyone could answer her, Akira joined the group and joined into the demonstration of the rules for Candace.

“But *what* is The *Question*?” asked Candace several times. We explained that the goal is to see whether any initial patterns could generate ever-expanding life. We disagreed about how to figure this out. A few of us wanted to take turns randomly setting up patterns and seeing what happens. We had done that for a while last week, until your article helped us to realize that a system starting with 1 creature might be good. You and Mr. Conway called them “counters.” Rodi calls them “creatures,” and we laugh when she forgets and refers to them as counters or stones.

Rodi was being a little pushy. We think she was getting a bit stressed to solve things since we only have one more class after today. She said, “Since we started to use this system last week, let’s give it a chance, and if you don’t want to keep going with it, we can change the way we’re doing it.” So we did. Rodi kept track of the results on the board. After about 20 minutes, a few of us got bored. Actually, more like frustrated. At this point, we had tested 7 set-ups today and some more last week, and none had continued growing after 2 or 3 moves. Rodi told our grownups that those 40 minutes of perseverance last week plus 20 today adds up to an hour of very impressive determination for kids our age. We had discovered that some set-ups lead to “death” (or what you call extinction), some to the “steady square” (or what mathematicians call a “block”), and some to the “infinity turn” (or what some of your friends call a “blinker” - a type of “oscillator”). We really enjoy naming these patterns! But so far, none continued growth.



We were excited to hear that in the olden days of 1970 you helped Mr. Conway offer a $50 prize to the first person to prove or disprove the conjecture that “no pattern can grow without limit.” We were wondering about too!

So anyway, Rodi admitted that this very boring approach to the problem does get tedious. (These days, Mr. Gardner, this type of problem is done on a computer.) We definitely would have had more fun randomly trying different arrangements/numbers of counters until we discovered our own method. Creating the set-ups from our imaginations instead of a rule is definitely more exciting. Is that what real mathematicians do?

At this point in class, two of us were talking about things that didn’t have to do with math. One hard thing in this math circle is that we REALLY like each other, but only see each other for this one hour per week. We want to socialize too!

Two of us were focusing on playing Life, and one was doing both.

Rodi said, “I think we’re ready to put our attention on something else.”

“I’m not,” protested Hudson.

“I know you’re not,” said Rodi. “I know that you could work on this problem all day long.”

“Yes I could.”

“Let’s try to come back to it later today if we have time.”

“Okay.”

So, we moved on. At this point, we have 2 working conjectures about Life:

1) There are no set-ups that can grow, starting with 4 creatures.
2) There is at least one set-up that will result in continuous growth.

Our group disagreed.

Okay, time to talk about something else:

MURDER…

…at the Ski Resort.

Since some of us were getting bored, we moved to a different table, and Rodi read us one of your “Tricky Mysteries.”

“A Chicago lawyer and his wife went to Switzerland for a vacation. While they were skiing in the Alps, the wife skidded over a precipice and was killed. Back in Chicago an airline clerk read about the accident and immediately phoned the police. The lawyer was arrested and tried for murder.

The clerk did not know the lawyer or his wife. Nothing he’d heard or seen made him suspect foul play until he read about the accident in the paper. Why did he call the police?”

We asked question after question after question, coming up with conjecture after conjecture after conjecture. After a few minutes, we realized that to test each conjecture, it was helpful to re-read the problem to see if the conjecture violated any of the rules. (Rodi was so impressed that we came up with this great math skill to on our own. C’mon, Rodi, we are ten already!) We discussed, while all Rodi really did was re-read aloud the problem, or parts of it, repeatedly, when we ordered her to.

Finally, Lily, Akira, Crystal, and Candace agreed on an answer: “*The clerk was familiar with the precipice, and knew that it wasn’t a dangerous place. Therefore the husband must have done something intentionally to make it dangerous. He pushed aside and piled up the snow to make her fall over the edge. Something in the newspaper article revealed specific info about the precipice, raising the clerk’s suspicions.*

Hudson disagreed. “How could you push aside that much snow?!” demanded Hudson. The rest of us came up with an answer that made him change his mind. We all agreed now. We looked at Rodi. “We came up with a solution,” said Candace. We demanded that Rodi re-read the problem to double check. We then agreed that nothing in our solution was wrong. Triumph!

“What does Gardner say the answer is?” asked Lily.

“Do you mean to ask what he says ‘an’ answer is?” asked Rodi.

“I mean, what does he think the answer is?”

Here’s your explanation, Mr. Gardner: “The clerk had sold the lawyer a round-trip ticket to Switzerland, and a one-way ticket for his wife.”

“Round-trip!” Crystal exclaimed. She jumped out of her seat. “That is so good!” She loved Gardner’s explanation. The rest of us were frowning.

“That’s it?” we asked.

“Yep,” Rodi said.

“Let me explain to you how round-trip tickets work,” said Crystal to the group. She thought (and was probably right) that not everyone understood. She explained, but the rest of us were NOT impressed.

“There could be many reasons that they weren’t planning to travel home together,” suggested several people. We talked about this. We decided that your “answer” left a lot to be desired. Rodi talked to us about the power of our logic – how causal assumptions can be knocked down/weakened with alternative explanations. She said the causal assumption here was the clerk’s, and possibly the police’s, assumption that if the couple went to Switzerland on vacation together, that then they would/should return together. We did not accept that assumption, and therefore we do not accept your solution.

“Ours is better,” said Candace. She was excited. We all nodded. Even Crystal was convinced.

Rodi was, of course, thrilled that we were so willing to disagree with such a famous smart person. Our open-mindedness is so inspiring to her. Her (boring) adult mind, of course, is too caught up in your fame. Rodi didn’t even think she would ever come up with a different solution to one of your problems, even though she knows people do all the time. That was one of the big points of your column in Scientific American, right? That plain old regular people like us might have a better idea than you did about how to solve a problem. We love this about you, and so does Rodi, even though she still feels intimidated.

Finally, we came back to your problem Scrambled Box Tops.

“I know the answer! Please, please, please let me say it!” begged Crystal right from the start. Rodi knew that Crystal knew a good answer because when we first tried this problem 2 weeks ago, Crystal saw the solution and said it right away out before Rodi could stop her. But last time, the rest of us were still trying to understand the question when she explained it. So no one knew what she was talking about.

But now everyone understood the problem. Lily and Akira were wiggling in their seats with solutions too.

Rodi said "I see you have an idea of what the solution might be, but so do some others. Let's make sure everyone gets a chance to state their conjectures."

"Let's raise hands," suggested someone.
"Good idea," said someone else.

Rodi said "If that's okay with the rest of you, I'll let Crystal go first and everyone will get a chance." The others agreed, as they could tell she was excited. Most of us raised our hands, so Rodi went around the table asking everyone for their conjectures. Here are our conjectures:

> *Crystal – If you pick a random marble out of the box labeled BW first, you can figure them all out from one.*
>
> *Lily – if you pick one out of BB or WW first, you can do it in two draws, and one if you're lucky.*
>
> *Candace – I don't know.*
>
> *Akiru – I can't remember. (He had lost his train of thought, even though he had an idea a couple of minutes ago. Frustrating. Rodi said this happens in math - Now you see it, now you don't.)*
>
> *Hudson – I want to see it! We could use items from the Life/Go set as props.*
>
> *Rodi said, Okay, let's do it.*

So we acted it out and agreed that both conjectures work. We were all satisfied and it was time to go home.

Sincerely,

Crystal, Candace, Akira, Hudson, and Lily

October 28, 2014, week 6

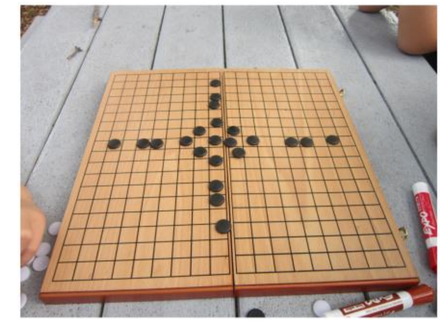
Dear Mr. Gardner,

Our last class was today. Mckenna wasn't here last week, so we began by explaining our problem to her: How can we figure out if any set-ups in Conway's Game of Life lead to a pattern of continued growth?

The method we started using is helpful. But it's boring! We looked at all of our results from the prior 2 weeks. "What should we do?" Rodi asked.

"Let's just do our own set-ups," said pretty much everyone.

Hudson had come to class with something he had been thinking about during the week – starting with a solid rectangle at the corner of the board. He wanted to try that. Akira wanted to test a 3x3 square that was empty in the middle. Seeing these 2 set-ups triggered Mckenna and Candace to want to test what happens when you set up "half the board" (a 4x8 rectangle). For the past 2 weeks, Lily had been asking whether it's possible to fill the whole board, but unfortunately, she wasn't here today. Crystal was a bit disappointed to not have a conjecture of her own, but she watched the others test some of these hypotheses. Four of us were working from 2 boards (one a Go board, the other Othello).



As we worked, people started throwing counters across the table occasionally. Rodi kept us calm with some of Mr. Conway's backstory (at our request). She read to us from a very entertaining article. Who knew that mathematicians could be interesting?

"Here's a photo of him. He's still alive, and he leaves nearby." Rodi told us.

Akira nearly leapt out of his seat. "Then there's still a chance that we could get the prize!" he said excitedly. Everyone wondered whether Conway was still offering that $50 that in 1970, you helped him offer to anyone who could prove or disprove that ever-expanding growth was possible. We discussed how little this prize was for such a famous magazine. We also wondered if we could visit Mr. Conway.

While we worked, Rodi told us more of Mr. Conway's backstory from the article. She thought we didn't notice, but we did notice that she stopped talking when the Life exploration got more intense, and then resumed the stories if anyone starting throwing stuff. Here are some of the things we talked about:

1) mathematical topics Mr. Conway is known for. We like the reminder that math is not simply arithmetic. We didn't know that one can be a "knot theorist." We also loved that Mr. Conway created a new number system, "surreal numbers." Four of us had created "Pumpkin Numbers" last year in our course about named number types.
2) a funny quote about Mr. Conway's enjoyment of teaching
3) Mr. Conway's Free Will Theorem (Rodi was so excited when she read about this; it changed the way she thought about things. But we kids were not in awe of this new idea at all since our thinking is fresh.)
4) Mr. Conway's opinion that the Game of Life is really "rather trite… trivial"
5) Mr. Steven Wolfram's disagreement with Mr. Conway about the relevance of Life
6) how/why Mr. Conway came up with Life – he was seeking simplicity
7) an expensive arithmetic mistake he made when offering a large cash prize. "And he's a mathematician!" said Mckenna. We were relieved that everyone makes mistakes.
8) why he suspects he may be a bad influence on people

9) his “recipe for success” in a mathematics career

Then Rodi told us something surprising about your friend Mr. Conway. “One more thing I forgot to mention from the article. He went to prison.” Some of our eyebrows shot up at that. Not Crystal’s though.

“Yeah,” she said. “He seems like the kind of person who would have gone to prison.”

“For what?” the others asked Crystal.

“For something really awesome!” she replied. Rodi quoted the article about Conway’s 11 days in prison for participating in a “ban-the-bomb” protest. This was exactly the type of “crime” that Crystal had in mind.

“No one would get arrested for something like that in our times,” said someone, but not sure. Then we argued whether it really “counted” as “going to prison” if he spent “only 11 days” there. We doubt you’ve been to prison. Have you, Mr. Gardner?

As we talked, we were making ground with our Life progress. With these new set-ups, things were neither stabilizing nor dying after just a few moves. The shapes of the populations were changing dramatically. We got a new idea: “If you fill the whole board or larger (rectangular) area, all but the corners will die and a new row will be born!” The half-board and open square that Candace, Mckenna, and Akira had started at the beginning had morphed into new shapes. Hudson’s smaller corner rectangle had gone extinct, so he started testing a new set-up.

More work, more stories. Candace and Mckenna got very excited. Their half a board had morphed into two “steady squares.” The conjecture morphed into “we’re done, it won’t die, but won’t grow anymore either.”

Then Candace asked hopefully, “Should we name it?” The others nodded. Rodi asked Akira for a name for his set-up; he called it “The Three-Square.” Rodi asked Candace and Mckenna- they named theirs “Half the Board.” Hudson didn’t name his.

While we were naming formations, someone noticed that the two steady squares on Candace and Mckenna’s board were just one diagonal unit away from each other. So a birth could occur! The end was not near, maybe. We went back to work, and Rodi to storytelling.

Then, sadly, the Half-Board set-up died after many moves. Candace and Mckenna decided to restart, from the position where 2 steady squares were one diagonal unit apart. As they worked that, we put our findings so far on the board:

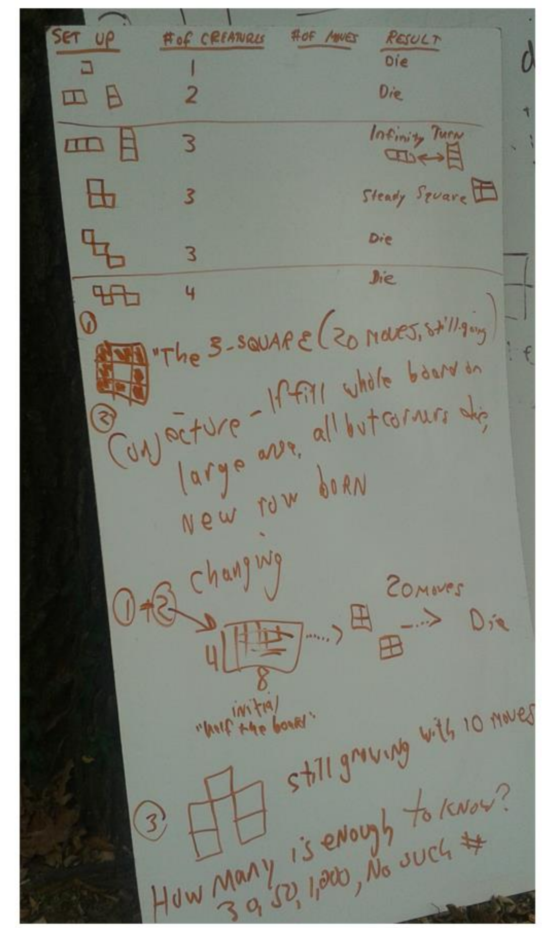
- The Three-Square: about 20 moves in, it was still growing
- Half-the-Board: died after 20-30 moves
- H’s second set-up: still growing with 10 moves

Then something really interesting happened. (For four of us, *all* of this was clearly interesting, we said “this is fun” more than once. For Crystal, it was not very interesting – probably because she didn’t have her own conjecture to test. Fortunately, she is a huge fan of biographies so thoroughly enjoyed the stories about Conway.)

Anyway, here’s the interesting thing we were about to mention: Candace and Mckenna’s new work with the 2 steady squares kept growing past the point where it did before. How could this be? Did we make a mistake? This set-up started in a different place relative to the edge, so does it matter where you place your set-up on the board? We wish you were here to work on this with us, Mr. Gardner. And that we had a larger board. Maybe an infinite board or a computer program.

Rodi asked us whether we could trust that any of these patterns that were growing at this point would continue to grow. We guessed how many moves would be enough to be sure:

- 30 moves?
- 50 moves?
- 1,000 moves?
- No such number/you can never be sure? (After we discussed this for about half a minute, everyone agreed with this conjecture.)



Since today was our last class, we discussed how to continue to play Life at home. Hudson said, “I want a Go board.”

As he mentioned last week, and the week before, Akiru said “I want to learn to play Go.”

"I can teach you," said Crystal, as she did every week. But Rodi never make time in class for this to happen. Then we came up with other ways to play Life at home if you don't have a Go board: Othello, Scrabble, possibly checkers, home-made with paper. (Some of these options, we thought, don't offer enough squares to progress very far.)

We had about 10 minutes left. The kids requested another of your "Tricky Mysteries." So Rodi read us

Funny Business at the Fountain:

> *"At a hotel in Las Vegas, a lady rushed out of the manager's office to get a long drink at the water fountain in the lobby. A few minutes later she came out for another drink. This time she was followed by a man. There was a mirror behind the fountain. When the lady raised her head, she saw that the man behind her had a knife in his upraised fist. She screamed. The man lowered his knife, and then both of them began to laugh. What on earth is going on?"*

Crystal immediately announced, "I have an answer!" A person working in the office was trying to sell a bunch of stuff to the couple at the water-fountain. It was really annoying. They wanted to get away from the person who kept trying to sell them stuff. They couldn't pull themselves away. So they faked the knifing attempt to distract the salesperson from the sales pitch. And it worked."

We had other ideas too, most involving some sort of prank. Since last week we solved the mystery by checking and rechecking the exact wording of the problem from the book, we did that again. We decided that we liked Crystal's solution best. Of course, we then demanded your explanation: *"The lady had the hiccups. Her boss was trying to stop them by frightening her."* Our solution is better. What we mean, Mr. Gardner, is that we all appreciated the logic in your solution, but just like last week, we appreciate our own solution more.

Then it was time to go. Thank you for giving us 6 weeks of math fun.

Sincerely,

Crystal, Candace, Akira, Hudson, and Mckenna

APPENDIX

ADVERTISED COURSE DESCRIPTION: Before there was Vi Hart, there was Martin Gardner. Celebrate the Martin Gardner Centennial with an exploration of Recreational Mathematics. For 25 years, Gardner wrote the Mathematical Games column in Scientific American, and became legendary for his unconventional approach to mathematics. In this circle, we will explore his life, his influence, and of course, his mathematical puzzles. The goal of this math circle is the same as the goal for all of them: to develop mathematical thinking. Recreational mathematics is yet another avenue for seeking patterns when none are obvious, and for seeking ways to crush seemingly obvious patterns that aren't really patterns at all.

ACKNOWLEDGMENTS

Thanks to Joanna Steinig, who was a participant in this course at age 10. Joanna is now 17 and helped reconstruct the narration here into student voices. (NOTE: All student names above are pseudonyms.)

Thanks to the Gathering 4 Gardner Foundation, for encouraging me to write this.

Thanks to Harmony Learning Community, which gave the Math Renaissance Math Circle a home for 10 years.

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AUTHOR'S NOTE

These letters are constructed from the following blog posts, which describe each session in detail:

https://mathrenaissance.com/martin-gardner-1-mutilated-chessboard-and-bronx-vs-brooklyn/

https://mathrenaissance.com/martin-gardner-2-switching-and-folding/

https://mathrenaissance.com/martin-gardner-3-maze-of-minotaur-and-scrambled-boxtops/

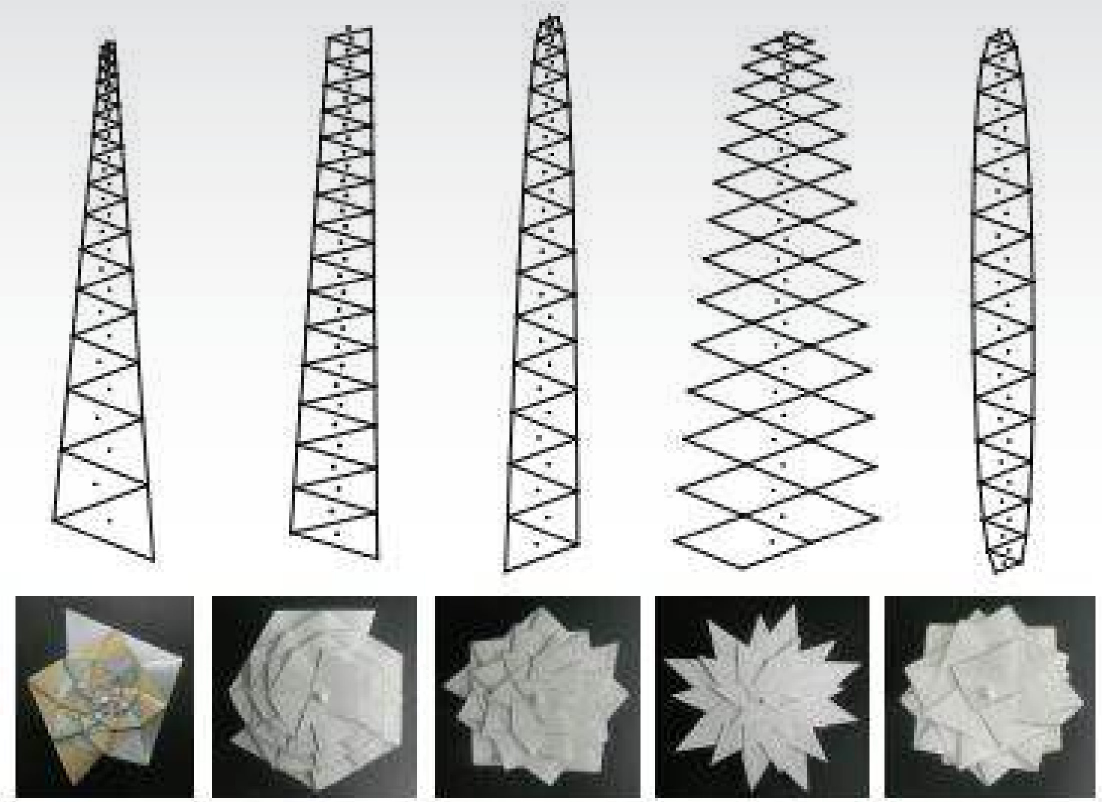
https://mathrenaissance.com/gardner-4-conways-game-of-life/

https://mathrenaissance.com/gardner-4-life-murder-and-box-tops/

https://mathrenaissance.com/gardner-6-more-life-and-besting-gardner-again/

CONTACT: rodi@mathrenaissance.com

SCIENCE



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Designing a flippe top

George Bell

September 2020

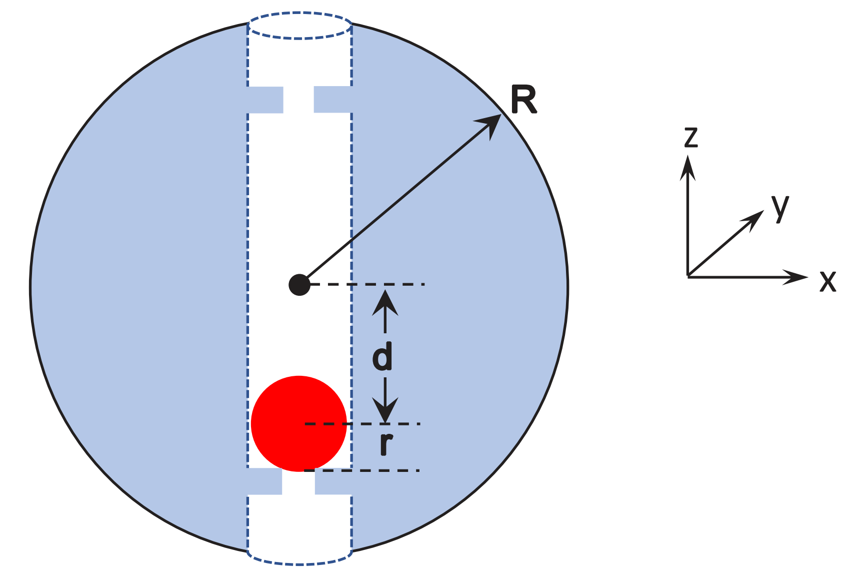

Figure 1: Diagram of a prototype flippe top.

Figure 1 shows a design for a new kind of spinning top. We start with a solid sphere of radius R. A hole of radius r is bored through the center and a steel ball (of radius r) is inserted. A stop at each end is necessary to keep the steel ball inside the sphere. The maximum distance the steel ball (center) can move away from the sphere center is denoted by d. At this stage there are only three design parameters: R, r and d, although the density of the materials used is also important.

Our goal is to design this top so that it behaves like a classical tippe top. When spinning at a sufficiently high speed, the classical tippe top is defined by two properties:

1) The Figure 1 position when the center of mass is directly below the center of rotation is ***unstable***.
2) The "inverted state" where the center of mass is directly above the center of rotation is ***stable***.

Suppose we can design the Figure 1 top with these two properties. When this top is spun about the z-axis, it is an unstable position and will invert. After inversion the steel ball will (presumably) drop down; the top is back at the starting position, but flipped 180°. This process will repeat, flipping over and over. Because this is a tippe top which flips repeatedly, I call it a "flippe top".

How can we determine if the top in Figure 1 will behave like a classical tippe top? Fortunately, several recent papers [1, 2] have considered the behavior of a sphere with an offset center of gravity. These papers give specific criterion for when the Figure 1 state is unstable and the inverted state is stable.

Material	Density (g/cm^3)
Hardwood	0.75
Stainless steel	7.8
PLA	1.25
PETG	1.27
ABS	1.07

Table 1: Top material densities.

We denote by I_x the moment of inertia of the top about the x-axis and by I_z the moment of inertia about the z-axis (we assume the top is symmetric in the x-y plane, so that $I_x = I_y$). The criterion for "tippe top behavior" is complex, but as an initial design target we want

$$I_z > I_x$$

All changes we make to the sphere can be thought of either as helping this inequality or going against it. For example, the steel ball increases I_x and only slightly increases I_z, so acts against this inequality. The hole through the center helps this inequality. This is one reason that we choose to bore the hole through the entire sphere. Such a large hole may affect the spinning dynamics when $r/R > 1/4$—in that case end caps may be necessary.

A wood flippe top

The easiest way to create a wood flippe top is to start with a solid wood sphere and cut a hole through the center. We then add a steel ball and a pair of stops to trap it. For stops one could use a pair of wooden washers. To determine good values for R, r and d we need to calculate the moment of inertia of this object.

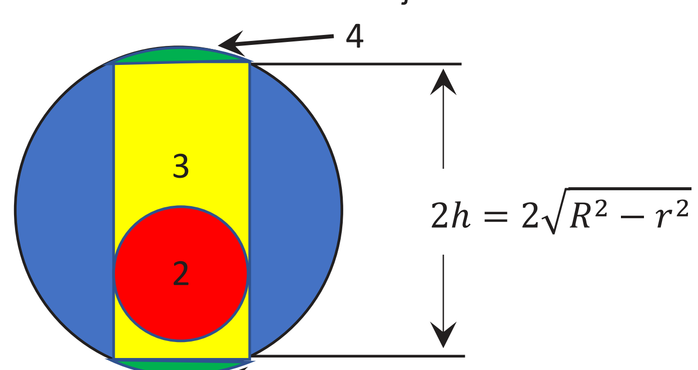

Figure 2: The four components of a wood flippe top. Component 1 is the full sphere.

Let us now calculate $I_x = I_y$ and I_z. Basically, we split the top into four components, shown in Figure 2. The mass and moments of inertia of each component are calculated, then added up to give the total mass and moments of inertia. If the mass of a component is negative it means that this component is removed.

Component 1 is a solid wood sphere with density $\rho_{wood} = 0.75\ g/cm^3$. The density of wood varies, but $0.75\ g/cm^3$ is a reasonable estimate for hardwood. I have many 1" diameter Maple balls which I weighed at 6.3 g each, this gives a density of $0.73\ g/cm^3$.

For component 1, we have

$$m_1 = \frac{4}{3}\pi R^3 \rho_{wood}$$

$$I_{1x} = I_{1z} = \frac{2}{5} m_1 R^2$$

Here m_i, I_{ix} and I_{iz} represent the mass and moments of inertia of component i. We will also use c_{iz} for the z-coordinate of the center of mass of component i, in this case c_{1z} is zero. In what follows, if c_{iz} is not defined it is zero.

Component 2 is the steel ball,

$$m_2 = \frac{4}{3}\pi r^3 \rho_{Fe}$$

$$I_{2x} = I_{2z} = \frac{2}{5} m_2 r^2$$

$$c_{2z} = -d$$

Component 3 (to be removed) is a cylinder of height 2h, radius r, and density $-\rho_{wood}$, where $h = \sqrt{R^2 - r^2}$

$$m_3 = -2\pi r^2 h \rho_{wood}$$

$$I_{3x} = \frac{m_3}{12}(3r^2 + 4h^2)$$

$$I_{3z} = \frac{m_3}{2} r^2$$

Component 4 (to be removed) are two cylindrical caps between z=–R and –h, and z=h and R where $h = \sqrt{R^2 - r^2}$. This component is very small when $r \ll R$ and can be ignored. This component is by far the most complex to calculate, but has almost no effect on the results. These formulas were verified using Mathematica. They also give the same values as the solid sphere when r=R and h=0.

$$m_4 = -\frac{2\pi\rho_{wood}}{3}(R-h)^2(2R+h) = -\left(m_1 + m_3 - \frac{4\pi}{3}h^3\rho_{wood}\right)$$

"napkin ring" term

$$I_{4x} = \frac{m_4}{20}\frac{16R^3 + 17R^2h + 18Rh^2 + 9h^3}{2R+h}$$

$$I_{4z} = \frac{m_4}{10}\frac{(R-h)(8R^2 + 9Rh + 3h^2)}{2R+h}$$

The total mass of the top is $m_t = \sum_1^4 m_i$. The center of mass of the top has z-coordinate ϵ, where

$$\epsilon = \sum_{i=1}^{n} c_{iz} \frac{m_i}{m_t} = -d \frac{m_2}{m_t}$$

By the parallel axis theorem, the total moments of inertia about the center of mass are given by a sum over the $n = 4$ components:

$$I_x = \sum_{i=1}^{n} I_{ix} + m_i (c_{iz} - \epsilon)^2$$

$$I_z = \sum_{i=1}^{n} I_{iz}$$

These values can then be used to create a phase diagram for the wood flippe top.

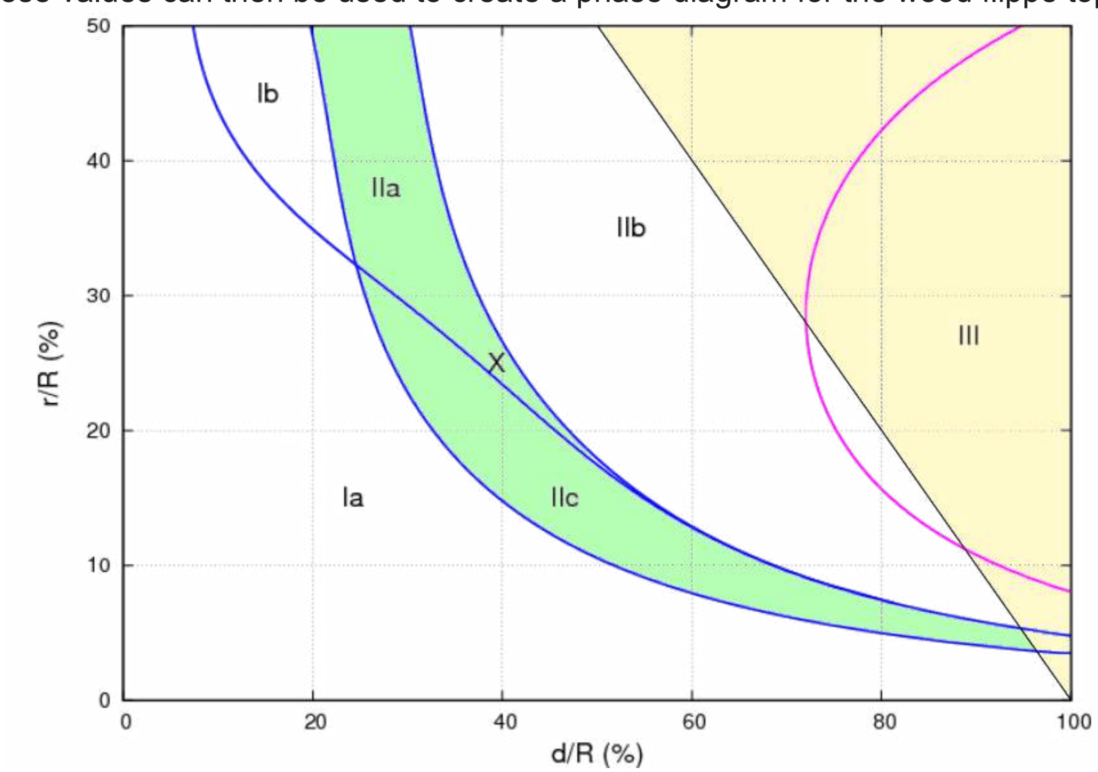

Figure 3: Phase diagram of the wood flippe top (r/R vs. d/R), see [1] for group definitions. A well-behaved flippe top must lie in the middle green strip.

Figure 3 shows the phase diagram for the wooden flippe top. The Roman numerals identify Groups as defined in [1], "classical tippe tops" lie in Group II. Tops in group IIb have stable "intermediate states", as defined in [1]. While these may not be a problem for a tippe top that flips only once, the flippe top must invert many times over a wide range of rotational velocities. We don't want the top to stop at an intermediate state, so a flippe top design should aim for Group IIa or Group IIc, the green region in Figure 3. For any choice of r/R this gives a narrow range of d values for the flippe top.

The yellow region in Figure 3 indicates $d + r > R$. These tops are problematic because the steel ball extends beyond the large ball and may interfere with spinning.

For example, suppose we start with a 2" diameter hardwood ball, R=25.4 mm and weight 51.5 g. A reasonable value for r/R is $1/4$ so we select a $1/2$ inch diameter steel ball. From Figure 3 we find that a tippe top with $r/R = 25\%$ lies in Group IIa or IIc when 28.4% < d/R < 41.4%. Thus, we should use a value of d between 7.2 mm and 10.5 mm, with d=10 mm marked by the 'X' in Figure 3.

One important parameter not shown in Figure 3 is the time it takes for the top to invert. We can infer, however, that the inversion time goes to infinity at the border between groups II and I, so we want to be at the high end of the range 28.4% < d/R < 41.4%.

Note that Figure 3 depends on the hardwood density used, as well as the density of the steel ball. If wood of a different density is used, or the ball is 3D printed, the curves shift to the right or left. The wood flippe top is scale invariant. If a wood ball of twice the size is used, it should behave the same provided all other lengths are also doubled (the steel ball must also be twice as large and eight times heavier).

A 3D printed flippe top

Objects 3D printed using an FDM (Fused Deposition Modeling) printer are generally not solid. It is faster to print the interior at a much lower density, the percent of interior which is filled is called the **infill rate** (f). In order to print faster this fraction is usually as small as possible. A typical value is f=15%, but in designing a 3D printed flippe top we can use any value between 10% and 100%.

Each layer needs to supported by the layers below, making a sphere difficult to print. One can add "support material" which is eventually discarded, but a better option is to cut the sphere in half and print it in two parts. Figure 4 shows a half-flippe top ready for 3D printing. After two copies are printed, they are simply glued together and/or held together with metal pins for alignment. The pins help hold the halves together, and also keep the central channel aligned. If you would like to print a copy, design files can be found at [3].

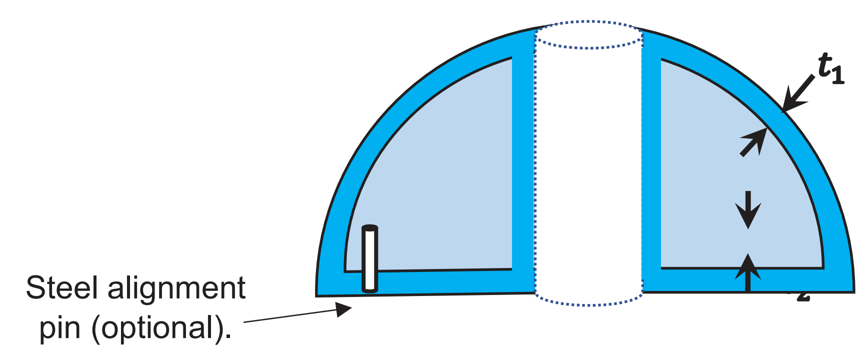

Figure 4: A design for half a flippe top for 3D printing.

In addition to R, r and d we now have two thickness parameters t_1 and t_2, as well as the fill ratio f. In general, if the ball is scaled up, the thickness parameters stay the same. For this reason, the 3D printed flippe top is not scale invariant, although it is approximately so.

We can calculate the moments of inertia of the top in Figure 4, but we now have nine components:

1) A solid sphere of spherical shell of density ρ. We will use $\rho = \rho_{PLA}$.
2) The steel ball of radius r and z-coordinate $-d$.
3) Remove a solid sphere of radius $R - t_1$ and density $-\rho(1-f)$.
4) Remove a cylinder of radius r, height $2(R - t_1)$ and density $-f\rho$.
5) Remove the "top lid", a cylinder of radius r, height $t_1$and density $-\rho$.
6) Remove the "bottom lid", a cylinder of radius r, height $t_1$and density $-\rho$.
7) Add back the "inner wall", a hollow cylinder with outer radius $r + t_1$, inner radius r, height $2(R - t_1)$ and density $\rho(1-f)$.
8) Add the "base plate" (doubled), a hollow cylinder with outer radius $R - t_1$,inner radius $r + t_1$, height $2t_2$ and density $\rho(1-f)$.
9) (optional) Add several steel alignment pins (see Figure 4). These hold the top together but also increase I_z.

We note that components 4-8 are not exact, but assume $t_i \ll R$. The phase diagram of the PLA tippe top with variable infill is shown in Figure 5. The f=100% top line in Figure 5 should correspond to the r/R=0.254 line in Figure 3. It doesn't match because PLA has a higher density than wood.

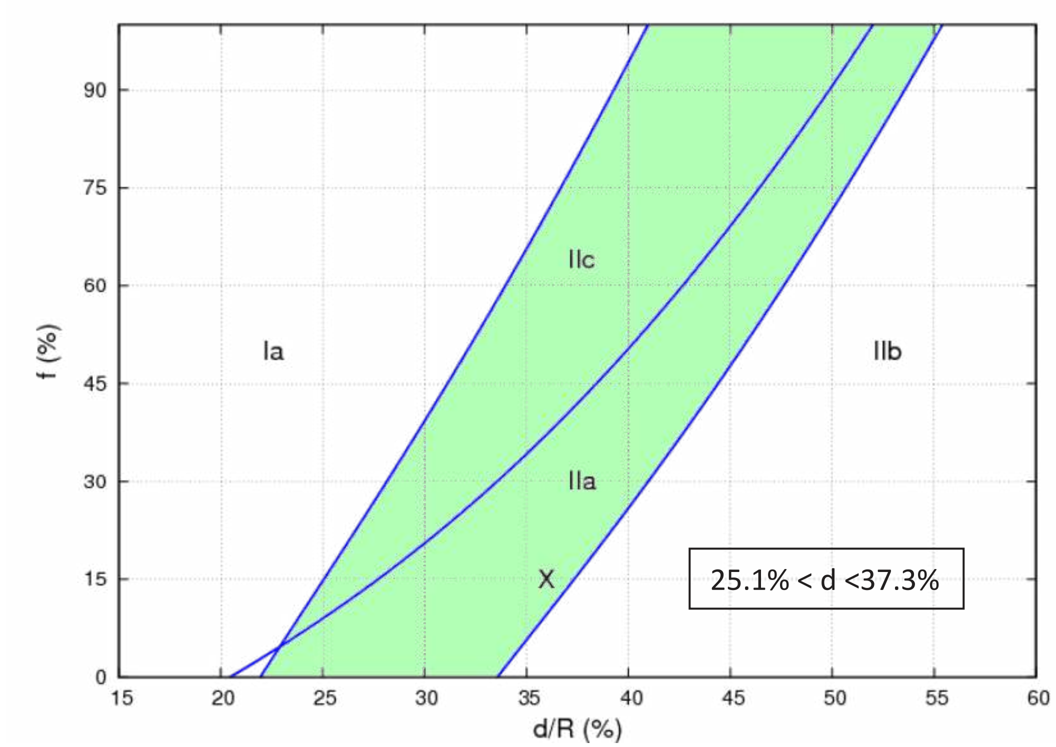

Figure 5: Phase diagram of the 3D printed flippe top (f vs. d/R), r/R=0.254.

I made many copies of the flippe top with the parameters in Table 2 (design files are available at [3]). In addition, I also scaled the entire top up by a factor of $5/4$ and down by a factor of $3/4$. All of them are able to execute multiple flips, as many as seven flips in the case of the smallest version. Figure 6 shows all three sizes.

Parameter	Value
R	25.0 mm
r	6.35 mm = ¼ inch
t_1	0.9 mm (2 perimeters)
t_2	1.1 mm (7 layers)
f	15%
d	9 mm (d/R = 36%)
ρ	1.25 g/cm^3 (PLA)

Table 2: Flippe top parameters.

Table 3 shows specifications for all three sizes of flippe top in Figure 6. The measured weight of each top is within 1 g of the Table 3 values, giving us confidence in our model.

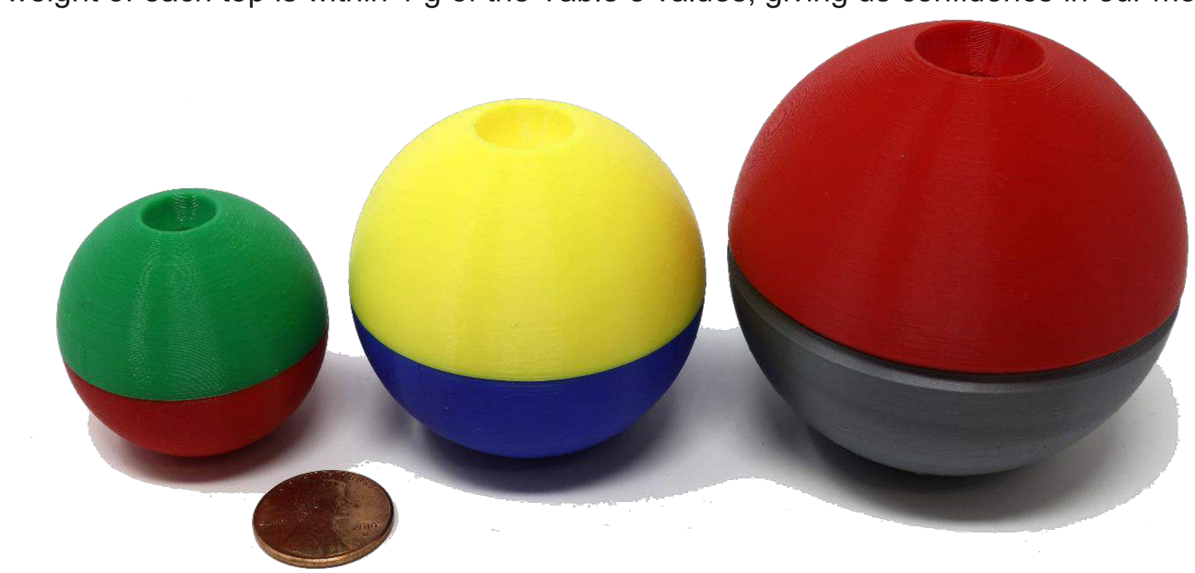

Figure 6: Three sizes of 3D printed flippe tops.
The largest one has an equatorial groove for a string.

Each time the top flips, some of its rotational kinetic energy is used to raise the steel ball by an amount 2d. Starting the top spinning by hand there is a limit to the initial velocity of the top, in the neighborhood of $v_{max} \approx 125\ cm/s$ speed of the top circumference. If all the top's rotational energy is used to raise the ball, the maximum number of flips which can be executed is

$$n_{max} = \frac{\frac{1}{2} I_z \omega^2}{m_2 g(2d)} = \frac{I_z v_{max}^2}{4 m_2 g R^2 d}$$

Inserting values for each size top gives the values for n_{max} given in Table 3. We find that smaller top and the wood top are capable of the most flips. I have uploaded two movies of tops in action on youtube [4, 5].

Parameter	Base model	5/4 scale	3/4 scale	Wood version
2R (ball diameter)	5 cm	6.25 cm	3.75 cm	2 inch
2r (steel ball diam)	1/2 inch	5/8 inch	3/8 inch	1/2 inch
d	9 mm	11.25 mm	6.75 mm	10 mm
m_2 (steel ball weight)	8.4 g	16.3 g	3.5 g	8.4 g
m_t (total weight)	32.6 g	58.5 g	15.7 g	55.1 g
ϵ/R (eccentricity)	9.2%	10.1%	8.1%	2.0%
$I_x = I_y$	73.0 $g\ cm^2$	199.6 $g\ cm^2$	20.3 $g\ cm^2$	124.6 $g\ cm^2$
I_z	75.5 $g\ cm^2$	204.4 $g\ cm^2$	21.4 $g\ cm^2$	133.3 $g\ cm^2$
printing time	2.8 hours	4.3 hours	1.5 hours	years to grow!
n_{max}	6.4	4.5	10.1	9.8

Table 3: Flippe top specifications for different sizes.

In order to get a faster initial rotation, a string can be wrapped around the top.

Observations of high-speed tops show a new phenomenon: the top performs one flip, but then the steel ball does not drop down (see the video [5]). The inverted top then spins until the rotational velocity drops below a critical value when the steel ball drops and flipping continues. It would be interesting to calculate this critical rotational velocity.

Summary

We have calculated design parameters for a (solid) wood flippe top and a 3D printed flippe top. I have printed many sizes of the 3D printed version, using many different values for d as well as the infill percentage f. The best versions use parameters given in Tables 2 and 3.

References:

1) M.C. Ciocci, B. Malengier, B. Langerock and B. Grimonprez, "Towards a Prototype of a Spherical Tippe Top", *Journal of Applied Math* doi:10.1155/2012/268537
2) M. C. Ciocci and B. Langerock, "Dynamics of the tippe top via Routhian reduction", *International Journal of Bifurcation and Chaos*, V 12, no. 6, pp.602-14, 2007.
3) https://www.thingiverse.com/thing:3990145
4) https://www.youtube.com/watch?v=aO6ZofZ5dos (movie of a small flippe top)
5) https://www.youtube.com/watch?v=3E_Ffhsj5mI (movie of a regular flippe top)

The Sirius Enigmas Mathematical Tops

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Abstract

I have developed a set of four spinning tops based on four of the most important mathematical constants [1]: ϕ, π, e and i. The tops are quite elegant, have different topological shapes and have unusual dynamical properties. Here, I discuss each of them separately, as well as a mathematical relation that unites them together.

PhiTOP® Development

In 2015, I did an experimental study of the problem of the rise of the center of mass of spinning objects, focusing on ovoids (eggs) and prolate ellipsoids. I presented the first results of this study at a meeting of the American Physical Society in April 2015. By the Spring of 2015, I had found experimentally that the "optimal" prolate ellipsoidal shaped object (optimal in the sense that it could be spun up easily, rise quickly and stand erect for a long time) has a ratio of major to minor axes of about 1.6. I decided to have such objects fabricated in various materials, chose the major to minor axis ratio to be equal to the "golden mean" ("golden ratio") ϕ ~ 1.618…, and named the resulting object the "PhiTOP®" or "ϕTOP®". The PhiTOP® was first presented in a paper entitled "The PhiTOP: A Golden Ellipsoid" at the Bridges conference on art and mathematics held in July 2015 [2]. More about the PhiTOP® can be found at: http://www.thephitop.com, in reference [3] as well as in the description in U.S. patent # 9,561,446.

PiTOP® Development

During 2016 and 2017, I studied the physics of spinning and rolling coins and flattened disks (right circular cylinders). As in the case of spinning egg-shaped objects, there is a lengthy literature of the physics of spinning coins that dates back at least to the 19th century. As with the PhiTOP®, I conducted a series of experiments to determine the "optimal" coin or disk (that is, one that spins and precesses for the longest time). The maximum duration spinning and rolling (precession) time was found to depend mainly on the ratio of the disk radius r to disk thickness t. I found experimentally that with r/t ~ 3 one produced the longest duration motion. I then chose r/t to be equal to π ~ 3.1415… and named the resulting object the "PiTOP®" or "πTOP®". Its volume V is exactly equal to r^3. The design that appears on one surface consists of a spiral containing the first 109 digits of π along with the Greek letter π in the middle. The PiTOP® was first presented at the "Gathering for Gardner" (G4G13) event in Spring 2018. More about the PiTOP® can be found at: http://www.thepitop.com.

Figure 1 (from left to right): *(a) brass ϕTOP®; (b) brass πTOP®; (c) brass iTOP; (d) brass eTOP.*

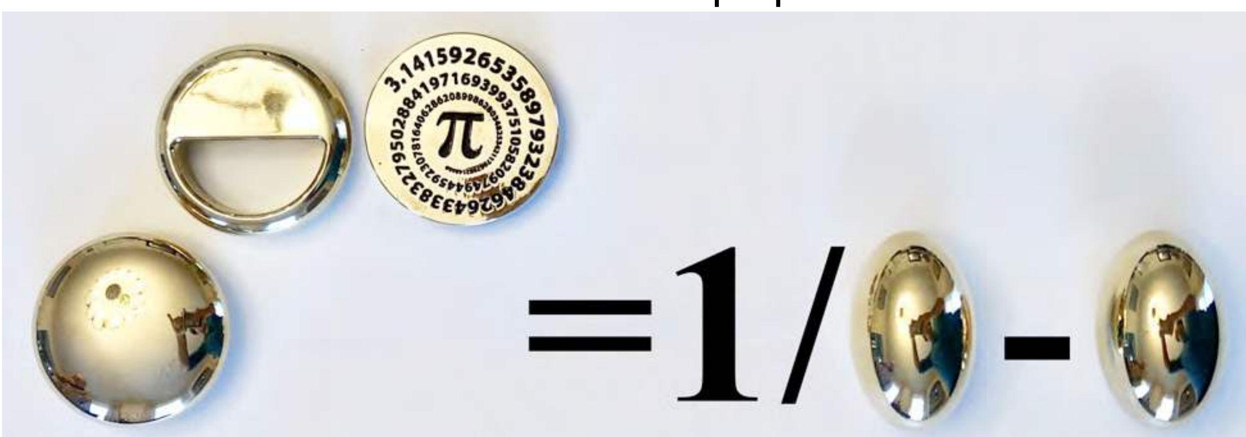
*e*TOP Background

Having designed both the PhiTOP® and the PiTOP®, I decided that the world also needed an "*e*TOP". In this case, no optimizing physics experiments were done. I simply chose to produce an oblate ellipsoid with the ratio of diameter d to maximum thickness t having $d/t = e \sim 2.718\ldots$ It can be spun like a coin. While spinning in its upright position, it presents a visible cross-section quite similar to that of the upright spinning PhiTOP®! As it settles down, it gives rise to beautiful Lissajous figure like reflections.

*i*TOP Background

Finally, I felt that the world also needed an imaginary top or "*i*TOP". After pondering the question of what such a thing might be and look like, I decided that a "real" imaginary top was an unlikely proposition (though check Nick Bantock's ideas about such things in his book "The History of Imaginary Spinning Tops" [4]). Therefore, I devised the "Inverting Ring Top", or "*i*TOP", a quasi two-dimensional "tippe-top" to join the other three "Sirius Enigmas" tops. By combining the definition of the golden mean $1/\phi - \phi = -1$ with Euler's equation: $e^{i\pi} = -1$ one has a relation connecting e, i, π and ϕ:

$$e^{i\pi} = 1/\phi - \phi.$$



Figures 2: A physical/mathematical relation between all of the Sirius Enigmas Spinning Tops.

Summary

In conclusion, I have devised a set of four unique spinning tops, each with its own novel physical and mathematical properties. Each has a longest dimension of 2 inches. When made in brass, each weighs between 4 and 8 ounces (though some of them have been produced in many other materials including aluminum, copper, titanium, bronze, stainless, steel, glass, plastic and wood). The four different types of "Sirius Enigmas" spinning tops can be thought of as being related through a mathematical identity.

Acknowledgments

I thank Boston University engineers Robert Sjostrom and David S. Campbell for their help in making the original metal versions each of the tops shown here. I also thank my daughter Kaz Brecher for her collaboration on the PiTOP® surface design and for the graphics in this presentation.

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Critical Point Conspiracy theories

As we enter the third decade of the 21st century, why – asks Robert P Crease – do conspiracy theories still abound?

Global warming is a plot manufactured by a global community of scientists. United Nations panels deliberately understate the radiation levels of the Fukushima and Chernobyl disasters. US media outlets contrive "fake facts" to refute Tweets of Donald Trump. Venal politicians are behind Ebola and other epidemics.

Groundless conspiracy theories are now an established feature of the political landscape. They resemble epidemics themselves, appearing from nowhere, spreading like wildfire, disrupting normal life, and being all but impossible to stop. They threaten democracy by poisoning the ability of voters to lucidly deliberate issues of human life, health and justice.

In her recent book *Democracy and Truth,* the University of Pennsylvania historian Sophia Rosenfeld argues that conspiracy theories thrive in societies with a large gap between the governing and the governed classes. Such conditions, Rosenfeld writes, allow some of the governed to reject the advice of experts as out of touch with "the people", and to create a "populist epistemology" associated with an oppositional culture.

Populists, Rosenfeld continues, "tend to reject science and its methods as a source of directives". Instead, such people prefer to embrace "emotional honesty, intuition and truths of the heart over dry factual veracity and scientific evidence, testing and credentialing". Modern science accentuates the gap between experts and non-experts, making it possible for populists to interpret "factual veracity" as tainted.

Shutterstock/Bernhard Staehli

Deep question Why do some people still think global warming is a conspiracy?

Galileo's gap

In my book *The Workshop and the World: What Ten Thinkers Can Teach Us about Science and Authority,* I argued that this scientific gap emerged with Galileo. Writing in his 1623 book *The Assayer,* Galileo used a striking image to defend his seemingly heretical studies of nature. The book of nature, he wrote, "is written in mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it".

The use of mathematics creates a rift between those unable to understand this special language and those who do, making it easy for the former to distrust the latter. Galileo's Gap, as I call it, has widened in size and consequence in the four centuries since then, feeding the frequency and severity of conspiracy theories.

Hard to believe, but I received hate mail after *The Workshop and the World* came out. Some concerned what I'd written about *The Preaching of St Paul* – a 1649 painting by Eustace Le Sueur that now hangs in the Louvre museum in Paris. This dramatic and imposing work shows St Paul looming above a pile of burning books, some with geometrical figures on their pages. The not-so-subtle intent was to portray heretics who read the book of nature as dangerous criminals.

Contemporary conspiracy theories, I wrote, show that St Paul is back.

My critics were furious. The painting is not about Galileo, they chastised me, but a passage in the Book of Acts 19:19, where St Paul's preaching prompted mystics to have "brought their books together, and burned them". Besides, the critics added, this issue can be settled factually by noting that the figures on the pages of the burning books resemble nothing found in maths texts. What's more, no trace exists of Le Sueur's intent, or that of the religious authorities who commissioned the painting. I must surely therefore be part of a conspiracy to slander the good saint.

I responded that of course the figures in the burning books were not in modern maths texts; they are what a religious firebrand of 1649 might think geometrical figures looked like. I also said that no factual information about the painting's creation could help us to understand its meaning, which can be understood only in the light of its historical context.

Le Sueur, a religious painter funded by church commissions, composed the work at a time when the most fundamental issue confronting the Catholic Church was that its claim to have the sole authority to interpret the Bible was being torpedoed by growing evidence in support of Galileo's mathematically based findings. Only that explains why a devout Catholic painter would devote enormous time and resources to create a 4 m high work about a handful of words in the Bible that mention book-burning – and then paint geometrical figures on the books' pages.

In a similar vein, the playwright Arthur Miller did not compose the 1953 play *The Crucible* because he had an interest in the Salem witchcraft trials. He did so to address the persecutions of supposed communist subversives taking place in the US in the 1950s. I probably did not convince my respondents. But their accusations that I had joined an anti-Christian conspiracy stopped.

The critical point

Modern anti-science conspiracies differ from their 17th-century antecedents, which emerged principally from the Church. Contemporary sponsors of conspiracy theories are multiple, spread not by preachings and paintings but by the Internet, and are energized by the ability to self-select information. But then, as now, conspiracy theories are not a sign of irrationality. Instead, they spring from the attempt by non-experts to make sense of often overwhelming and contradictory information based on personal values, available evidence, whom one trusts, and experience.

To reduce the impact of conspiracies, there's little point quoting mainstream experts, citing scientific papers, appealing to facts, or even teaching more science, for all these things will be said to belong to the conspiracy. Far more effective is to provide people with better tools to make sense of their personal, political and social experience. Yet the disciplines that cultivate these interpretive tools, collectively called the humanities, are largely having their resources redirected to the sciences.

Ironically, the dazzling and visible successes of the 21st-century sciences are overshadowing and undermining the 21st-century humanities that ground the authority of the sciences themselves.

Robert P Crease is chair of the Department of Philosophy, Stony Brook University, US, e-mail robert.crease@stonybrook.edu. His latest book is *The Workshop and the World: What Ten Thinkers Can Teach Us About Science and Authority* (WW Norton)

FIBONACCI TURBINE and Cone-puter for Cone-tinued fraction by Cone-pass

Akio Hizume

Geometric Artist

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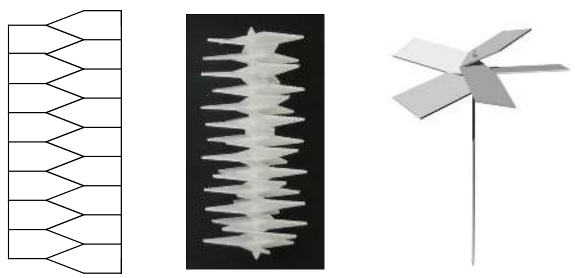
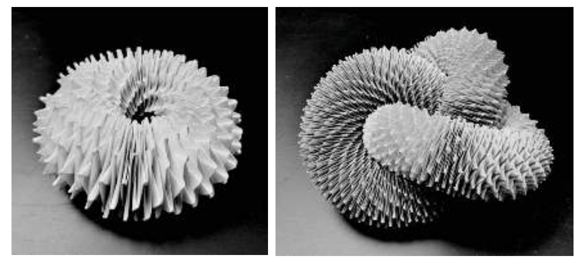


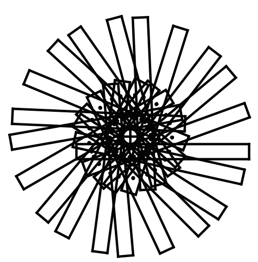
FIBONACCI TURBINE and FIBONACCI HELICOPTER

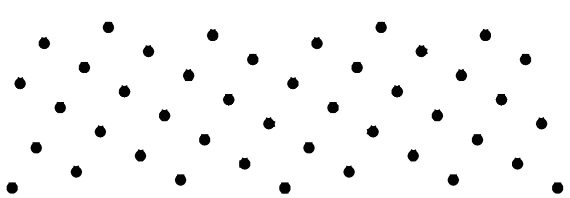
Just after G4G13 (2018), I invented the ORIGAMI Fibonacci Turbine (Patent No. 7013094)[1].
I have been working with the Phyllotaxis (Golden Angle) for 30 years, and this is my latest work.
It is easy to make like ORIGAMI. There is no shaft, so there is no weak. Any blade is not on the same plane, the loss of lift is low.
The actual spin and flight can be seen below.
https://youtu.be/8naOjOPWK5Q



FIBONACCI TORUS and KNOT (2019)

Make long Origami Fibonacci Turbine and join the ends to form torus or knots.
They have a very unique rotational movement.
https://youtu.be/LdnvxN4UUfs

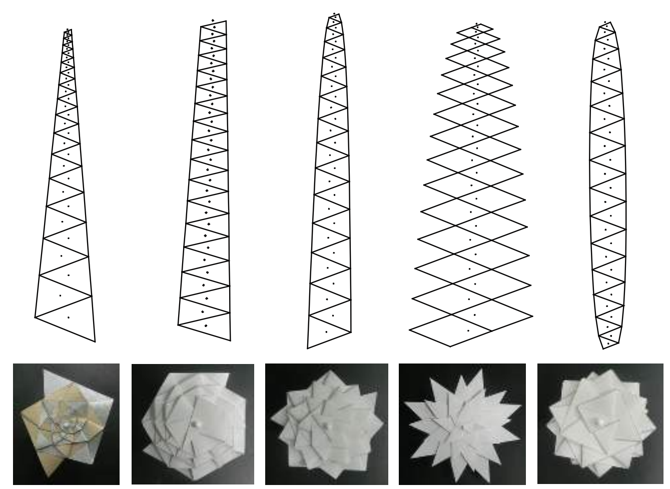




FIBONACCI WHEEL

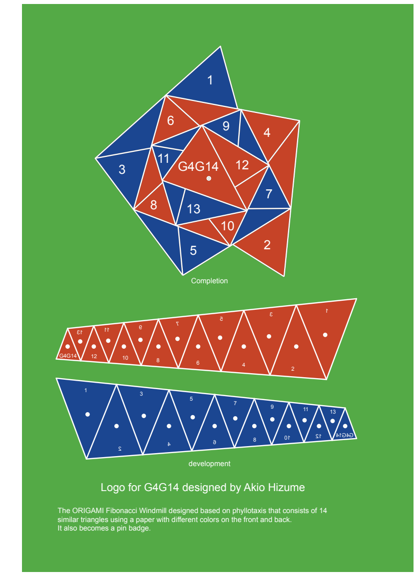
I made the Fibonacci Turbine of bamboo.
It rolls well without circular tyres.
I name it as Bamboo Fibonacci Wheel.
https://youtu.be/4wc0qpLRuH8

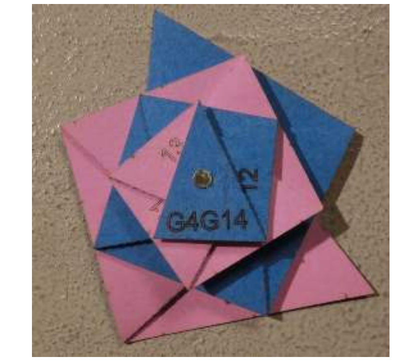
The wheel's footprint can be used as a musical score to play four different types of music related to quasiperiodic music.
https://youtu.be/8-kWbufOPsg



GENERALIZED FIBONACCI TURBINE

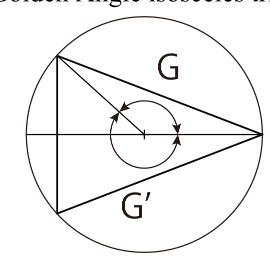
Origami Fibonacci Turbine can be transformed into any curved surface, including cone, parabola, catenary, hyperbola and ellipse, etc...
In addition, turbines other than golden angle can be freely designed based on any real number of its continued fraction.
In particular, the use of parabola Fibonacci Turbine reproduces plant petals excellently.





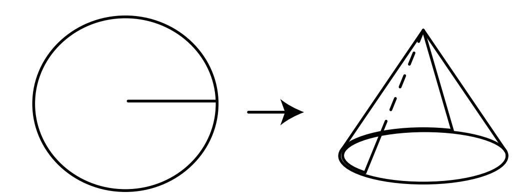
These are some variations of the Fibonacci Turbine.
They are distributed as my G4G14 Exchange Gift.
This pin badge is reversible.

Golden Angle isosceles triangle



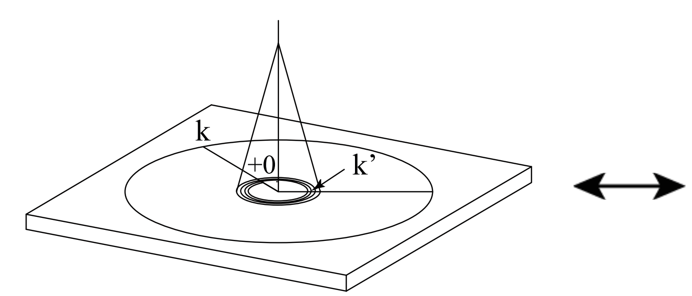
$$G = 2\pi(2-\tau)$$
$$G' = 2\pi(\tau-1)$$
$$\tau = \frac{1+\sqrt{5}}{2}$$

Cone-pass



CONE-PUTER for CONE-TINUED FRACTION by CONE-PASS

The basic figure composing the Fibonacci Turbine is the Golden Angle isosceles triangle.
It is impossible to construct the Golden Angle using only a ruler and compass, and it has been a long-standing question how plants acquire the Golden Angle.
In 2020, I found that the Golden Angle could be constructed exactly by the operation of making a cone from a circle with a slit in it, which I named the "Cone-pass" as a new tool for construction[2].
In 2022, it was found that this method can be used to construct not only Golden Angle, but also any real number of angles.
It was also found that the method is closely related to continued fraction.
The preprint saved to Reseachgate[3] will be published in the G4G14 Exchange Book.



$$x_{k+1} = \frac{b}{a + x_k}$$

$$x_\infty = \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cdots}}}}$$

$[\bar{a}]_b$ continued fractions

Cone-tinued fractions

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Does Conway's "Game of Life" predict that the speed of light is constant in the real world?

"Gathering 4 Gardner 14" presentation

10 April 2022

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The kind of feedback I need: Please tell me whether these ideas are (1) wrong, (2) true but trivial, (3) good for explaining concepts in textbooks, (4) new and novel but not very important, or (5) new, novel and important. Thank you!

Abstract

Conway's Game of Life needs no introduction to fans of Martin Gardner.

Numerous examples exist in which a GoL "spaceship" object sends out smaller objects ("gliders", etc.) that move away from the generator at a constant speed. That speed is set by the underlying nature of the GoL's generations, because each such object can move no faster -- generation to generation -- than the unit distance built into the GoL model itself. In fact, the maximum such speed of propagation is called *c* or "the speed of light" in GoL terminology.

In a typical computer running a GoL in a lab, the unit of time in the GoL of course has no fixed relationship to the unit of time marked on the lab's clock on the wall. If the lab personnel were to double the rate of GoL generations (relative to the clock on the wall), the speed at which a GoL glider moves across the screen is also doubled (relative to that clock), but within the GoL universe itself, there is no change in speed.

Might this phenomenon actually apply in the real world of spacetime, and its actual speed of light?

The very concept of spacetime encourages us to think of time as a literal dimension. If so, what is the multiplier used to convert units of time to units of distance? The natural conversion factor would be to set the unit of time to be the Planck time, and to set the unit of distance to be the Planck distance; the conversion factor would be the speed of light.

Now imagine that the universe is the surface of some hypervolume, which is expanding -- being laid down -- along the time dimension. The "future" is akin to some kind of gas: an energy field with no structure or organization. The "past" is akin to a solid structure that has previously condensed out of the gas. The "present" is the condensing layer between this past and the future. The present is a layer of spacetime, which contains matter in particular positions. The layer also has waves that can transmit energy by moving in the various spatial directions.

A single assumption leads to the prediction that the speed of light in the real universe is a constant, just as it is in the GoL. The assumption is that a particle in layer t (for present time), at position coordinate X (in any given spatial dimension) can at time $t + 1$ be positioned at X, X – 1, or X + 1. A bit of the energy from the future "gas" can condense into the layer of the present moment only by latching onto a particle in the Present layer that is positioned at, or next to, the point at which the condensation occurs.

If we were to imagine that this hypervolume exists in some uber-being's computer lab, for those of us living in the hypervolume, it doesn't matter how fast the Planck-thickness layers accumulate.

If light is something that undulates on the surface of this expanding spacetime solid, then it can propagate through space no faster than it can move from layer to layer of time. Each additional time layer permits no more than one Planck step in any spatial direction. Regardless of whether we regard light as a wave or as a particle, its angle through spacetime can never be more than 45° away from the path of a stationary object.

Finally, the model sheds light on the wave-particle duality nature of matter and energy. A "particle" is a kink in the outermost layer of spacetime that stays in place, or moves one unit away, as waves and events in the condensing layer progress. In the condensing fluid, such a kink is a wave; once condensed, it's a particle whose location can be known to within a Planck length.

Introduction

For several years, the consensus of astronomers has been that the universe is, in the present era, expanding at an accelerating rate. This consensus was based on well-founded empirical data that seemed to compel such a conclusion. A report by Milne et al. (2015) concluded that the universe's expansion is slower than this previous consensus suggested. If this revised conclusion is correct, then now may be a good time to consider models that feature or require a constant rate of expansion of the universe.

If a universe expanding with zero acceleration is indeed consistent with observation, then some simple postulates about the nature of time and the fabric of spacetime lead to answers to two commonly-asked questions: (1) Why is the speed of light a constant? and (2) Why can nothing travel faster than the speed of light (or, as will be argued, the speed of gravitational waves)? This paper explores an extremely simple model of an expanding universe that straightforwardly yields answers to these two questions.

As it happens, John Conway's Game of Life (GoL) also has a maximum travel speed —for reasons that are very similar to the reasons proposed in this paper for the real world. The purpose of this paper is to set out this theory of the universe, and use the GoL as an illustration of many of its principles.

This model has several fundamental predictions. It suggests why time is the physical parameter that we can measure with the greatest degree of accuracy, and explains how the forward arrow of time is inextricably linked to the expansion of the universe, why the rate of expansion seems to be exactly what is needed to avoid collapse, why spacetime is so exceedingly geometrically flat (without needing to postulate a special mechanism for a period of hyperinflation), and why light cannot complete a circumnavigation of the universe. Moreover, the model predicts that the universe is prevented from collapsing back into a singularity by virtue of a mechanism that would emerge in equations as a parameter related to dark energy. Finally, it explains a fundamental difference between hadrons and bosons, and may incidentally explain why matter predominated over antimatter in the early universe.

John Conway's "Game of Life"

There is an earthly universe in which there is a maximum speed of travel—a speed that follows from the laws that underlie the passage of time in that universe. That is John Conway's Game of Life. In a GoL universe, each particle at any given point in time contributes to the creation of a particle at adjacent points in the next point in time. GoL fans are familiar with arrays of points that generate, and/or propagate, so-called gliders that (when animated) move through GoL space at a fixed maximum speed. No GoL object can travel faster than this, because the particles that make up the glider at time t

can only influence slots in GoL space at time t + 1 that are one pixel away from it at time t. Indeed, this speed, in GoL terminology, is called the "speed of light".

The purpose of the present paper is to explore a model of the expanding universe that has a maximum speed that is almost perfectly analogous to this mechanism. Nothing can travel faster than the speed of light in our universe because each moment in time is a layer in an expanding universe, and each bit of matter in layer t of that universe can move no more than one layer's worth away in layer $t + 1$.

The Postulates

The theory follows logically from a small number of postulates about the nature of spacetime. Many of these have been previously posited (and long ago) by other scientists. However, one key postulate (number 4) appears to be novel. In combination with the others (especially postulate 5), it explains the constancy and the maximality of the speed of light.

• Postulate 1: Space and time are inextricably united in a fabric-like substance called spacetime

This is, of course, the point of view of Einstein's special theory, Minkowski space, and the Lorenz transformation equations, and needs no elaboration here. Spacetime's fabric-like nature is likewise widely appreciated.

• Postulate 2: The time dimension is exactly like the spatial dimensions, yet is different

This is also generally appreciated, but a certain aspect of it needs emphasis. Like many others, the proposed theory presumes that spacetime is a real thing—not merely a useful analogy—and that the time dimension is in some ways exactly like the space dimensions in a deep, tangible, and probably literal way.

To fully appreciate the model, the units used to measure space and time should be identical. But this principle (also a familiar one) goes beyond the use of a distance measure such as light-years. One ought to be able to say either that "the ball moved 300,000 km to the north" or "the ball moved one sec to the north". Likewise, one should be able to state that a ball that appears to us to be at rest in space either "moved 1 sec into the future" or "moved 300,000 km into the future".

This postulate is scarcely controversial in and of itself. The key is that it encourages us to equate (in Postulate 4) the speed of light moving through space with the speed of objects moving through time.

• Postulate 3: Planck distances and Planck times have a fundamental reality

Something fundamentally important (and very strange, from the point of view of our limited experience of the macroscopic world) takes place at Planck times and Planck distances. The time it takes light to travel a Planck distance is a Planck time (in accord with Postulate 2). If there is any granularity, quantization, or fundamental minimum distance in the structure of space (and by extension, the measurement of location), there is an equivalent granularity, quantization, or fundamental minimum time—and vice versa. If space and time are one, then so are Planck distances and Planck times.

• Postulate 4: The expansion of the universe is—***is***—the progression of time, and thus must always be experienced as constant by objects in spacetime

This is the key assumption. Like sheets of paper accumulating into a ream, like acetate cels piling up to create an animation or flipbook, like a mineral seed planted in a hypersaturated medium growing into a crystal, like a blob of gelatin slowly solidifying as it cools into a bigger blob of gelatin, like a layer of living coral (on the surface of a growing ocean floor) that builds its present life on its dead calcium skeleton below the surface, the expansion of the universe into the time dimension *is* time. The universe as we experience it consists of the outermost Planck layer of that crystallized product, a layer that is constantly changing as it grows. The passage of time is the experience of motion in the direction of the time axis, with objects as we experience them at each moment existing only in that Planck-thickness layer. The universe expands by accumulating layers, each of one Planck time/distance in thickness.

Someone at rest in our universe, but by necessity moving along the time axis, can observe and measure oscillations in local objects; we call those oscillations "time". If someone moves along a space dimension (also through time), they travel on a diagonal path through spacetime, but they will still observe and can measure oscillations that they will experience as "time". And as far as they are concerned in their frame of reference, their gelatin-surface world is still growing (i.e., moving through time) layer by layer, with each layer being just as thin and Planck-like as any other. Thus, an object moving along a diagonal spacetime path is experiencing the expansion of the universe at the same constant rate as everyone else.

In this view, it is impossible (nonsensical, actually) to observe that the universe is expanding at an accelerating rate. Imagine being a cartoon character who exists by virtue of being a drawing on an acetate cel, and imagine a clock on the wall of that character's environment. The godlike creatures that produce the cels, stack them, and flip through them to create an animation can produce each cel on their own schedule in what might provocatively be called godtime. It is irrelevant how long it takes for these

reasons that are unclear) the notion that light is also some kind of undulation in the fabric of spacetime (Einstein, 1920), even going so far as to use the word "ether" in connection with that fabric. Of course, the idea of a luminiferous ether is inconsistent with observation, but those famous observations pertain to the hypothesis that "ether" is a substance that exists *in* spacetime. To the extent that Einstein's theory (and the present theory) include a concept that resembles ether, it is not *in* spacetime: it *is* spacetime.

Instead of saying that the speed of light is a speed limit for everything in the universe, it would be more primary to declare that the speed limit is the speed of gravitational waves. Logically speaking, the insight that nothing travels faster than the speed of light is secondary to the insight that nothing travels faster than the propagation of gravitational waves. It follows trivially that light does not travel faster than that speed.

However, there is a sense in which objects *can* travel at the speed of light—and always do. An object may be at rest in the three dimensions of space, but that object is nevertheless traveling through spacetime. Indeed, the point of view of the present theory is that all objects travel through time, literally, at the speed of light.

At the risk of being overly epigrammatic, it might be said that space travels through time at the speed of light, and light travels through space at the speed of time. These are merely restatements of the postulate that space and time must be measured with the same units.

• Postulate 7: Hadrons are movable, semi-stable structures in the fabric of spacetime

A simple knot loosely tied in a rope can be moved from point to point along the rope by inserting one's thumb into the loop of the knot, moving one's hand left or right, and allowing the rope to slip around the thumb (taking care to avoid pulling the knot tighter). Because the word "knot" has a very specific topological definition that is contrary to this lay concept of a knot, I will use the word "kink" to refer to the concept of a knot-like disturbance in the (otherwise) smooth hypersurface of the expanding universe. Although the present theory is agnostic on the question of the reality of strings, aficionados of string theory are welcome to think of kinks as being composed of vibrating strings.

Kinks can be inside or outside of the expanding surface of the fabric. On the outside, they are postulated to be matter; on the inside, they are antimatter. When they collide, they annihilate each other (by unkinking each other, symmetrically) and release the potential energy they had stored. By their nature, even when motionless, they disturb and distort what would otherwise be a perfectly smooth, expanding universal hypersphere. Perhaps this distortion (like that of a kink in a rope) is smoother when viewed from one side than the other.

When a kink moves, it carries this distortion with it. To the extent that this distortion occurs along the time dimension, a moving kink resembles a wave—albeit an irregular one, and one that does not have a tendency to propagate in all directions (as most waves, such as sound and light, do).

• Postulate 8: Gravity is the force that glues one layer of spacetime to the next, by virtue of its strength at Planck distances and in the time dimension

Gravity is often described as the weakest of the fundamental forces of nature, but gravity is not a weak force at Planck distances (Physics Stack Exchange, 2015). The other fundamental forces (electromagnetic, strong, and weak) behave according to equations that can operate only on the surface of expanding spacetime. Only gravity's equations permit it to exert force from one layer to another. The theory postulates that a particle (kink) that exists in layer t of the expanding hypersphere attracts condensation (of some sort, in some sense) in layer t + 1, and that the two particles are bonded to each other by the force of gravity otherwise operating outside of our awareness.

• Postulate 9: Hadrons and bosons are fundamentally dissimilar

According to Postulate 7, hadrons are particles, and are localized deformities (kinks) embedded in the surface of spacetime. Thus, when they move, they have some wavelike properties, but they are not fundamentally waves. Kinked, they contain potential energy. Unkinked, they release that energy, which can then be re-kinked or completely released as moving undulations.

In everyday experience, a fundamental property of what we call "particles" is that they can move or they can stand still. You can hold them in your hand, or you can throw them away. If they stop moving through time, they disappear; we say they are destroyed, or converted into pure energy.

How can you hold a moonbeam in your hand? You can't. Bosons, by virtue of their wavelike nature (Postulate 6), *must* move through space. If bosons stop moving, they disappear; we say they are absorbed. Moreover, applying the time dilation equations to bosons results in the conclusion that bosons do not experience the passing of time. Indeed, their motion across the hypersurface of the expanding universe does not leave any tracks, and in that sense they are timeless. Particles, on the other hand, leave a record of where they have been (see Postulate 10).

Again at the risk of appearing overly epigrammatic, note that hadrons must move through time but need not necessarily move through space, whereas bosons must move through space but do not experience the passing of time.

• Postulate 10: The universe began not with a big bang but with a little plop

The simplest hypothesis about the earliest moment in the history of the universe is that it began as something no larger than an entity of Planck-sized dimensions—a seed, if you will. (This is termed the Planck epoch.) The present theory imagines that the universe then expanded as a 4-dimensional hypersphere around this seed, and that (as previously mentioned) our experience of the universe consists of the 3-dimensional surface of that hypersphere combined with a perception of the passage of time. Each unit of matter on that surface attracts another unit of matter that "condenses", in a sense, from the formless energy soup of the surrounding future. One can think of this energy soup as a field: perhaps a Higgs field.

Like a special type of coral that lives and grows atop an expanding ocean floor, the universe's present state is displayed on the surface, and its history is recorded beneath the surface. The cosmos is, in this view, the surface of a kind of crystal—a solid (or at least non-fluid) structure that possesses a particular past and an uncertain future.

Whether this imagery can be converted into equations that can be tested against experiment is an exercise for the future.

Deductions

Accuracy of time measurement

If the progression of time is by its nature constant (Postulate 4), it might make sense that the measurement of time is the physical parameter that can be measured to the highest degree of precision. This seems to be the case. Lombardi (2002, figure 17.1) indicates, for example, that seconds can be measured to an accuracy of 10^{-15}, whereas the next most-precisely-measurable quantity is length (to 10^{-12}).

The rate of expansion appears to be precisely regulated

If the rate of expansion of the universe constitutes time, there need be no puzzlement about why the universe's expansion seems to be *precisely* what is needed to avoid eventual gravitational collapse. It also explains why the universe has not already expanded so quickly that stars could not have formed and entropy would be the universe's most obvious property. There is no need to invoke the anthropic principle or multiple universes—or godlike creatures.

Dark energy (but not necessarily dark matter?)

Dark energy is postulated as a force that causes matter to repel other matter, and thus counteracts what has been thought to be a natural tendency of the universe to contract as a result of gravitational attraction. In the present model, there is no need for such a specific force operating in the spatial dimensions. What keeps the universe expanding is the fact that the layer of the hypersphere created at time t + 1 rests, almost literally, on the layer created at time t. The matter in layer t triggers the condensation-like creation of layer t + 1. Layer t continues to exist forever. According to Postulate 8, gravity itself is the force that binds matter in each layer to matter in subsequent and preceding layers. The word "bind" correctly suggests an attraction, but the idea of a repulsion is implicit in it; two particles that are glued together cannot separate, but they also do not collapse into one.

If one were to write equations that describe this condensation/accretion process, dark energy might emerge as a parameter or variable that appears when such equations are restricted to the domain of the expanding hyperspherical surface.

Dark matter, on the other hand, does not specifically appear in the present theory. Whether or not it exists is not addressed by the theory. Either alternative is possible.

The predominance of matter over antimatter

If an inflated balloon is coated, inside and outside, by a layer of particles of uniform size, slightly more particles can be fitted onto the outside than the inside of the balloon. If spacetime has (or had) a positive curvature, might this account for the predominance of matter over antimatter? This deduction seems likely in the light of Postulate 7. If there was a time in the history of the universe when spacetime was positively curved, and if a highly energetic process generated kinks on the outside and inside of this hyperspherical surface up to their respective volumetric limits, somewhat more kinks would be created outside than inside. If, as the hypersphere grows and its contents cool, the indiscriminate generation of kinks were to cease, then those inside and outside of the surface would annihilate each other. The result would be a universe in which the "matter" kinks (outside) numerically exceed and destroy the "antimatter" kinks (inside) —but only by a small percentage.

Photons are their own antiparticles

Besides being a known fact, this follows from the nature of waves. If a photon is an undulation on the surface of the expanding hypersphere, then its antiphoton would be an undulation of exactly the same wavelength and amplitude, but of opposite sign: undulating out when its photon is undulating in, and vice versa. By the nature of waves, this is just another undulation: one that is 180° out of phase with the original.

Photons move forward and backward in time—just a bit

Sound is a wave whose undulations take place in the direction of the sound's motion. A vibrating rope is a wave whose undulations take place in spatial dimensions perpendicular to the direction of the wave's motion. In what direction do light waves undulate?

The displacements of the undulations of a propagating light wave are in the direction of advancing time—i.e., minuscule hills and valleys wrinkling and moving across the hyperspherical surface. In a sense, then, every wave of light moves ever so slightly forward and backward in time as it rides the hypersurface of the expanding universe. More precisely, from moment to moment it is just a little bit ahead of the average advancing surface and then just a little bit behind it.

Is a special mechanism for hyperinflation necessary?

As the hypersphere expands in the time dimension, it necessarily also increases in diameter in the spatial dimensions. Each layer of time is the thickness of one Planck unit, and like any regular object that is circular in cross section, it has a circumference of 2π times the radius. The early universe was thus not only much smaller, but also of much higher curvature than it is today. But because a Planck-sized unit of space is of finite (albeit minuscule) size, the curvature of a one-Planck unit of space (in the Planck epoch) is not infinite—although it would have been extremely high.

Although the proposed theory is consistent with a period of early hyperinflation, there is no need to postulate a *special* mechanism to drive a hyperinflationary interval near the beginning of time. If the size of the universe is considered to be the circumference of this hypersphere (as opposed to a surface area or a volume), the circumference of the universe doubled in the first Planck time of its existence (an increase of 100%, as it grows from a radius of 1 Planck length to 2, in the Planck epoch). With the addition of the next layer, it increased 50% more (from 2 lengths to 3). Subsequent layers increased it by 33%, 25%, 20%, and so on—a series that, by the time it reaches the present era, has a rate of increase that has for (more than) all practical intents and purposes been constant for a very long time—a very low number, not actually zero, but nearly so, and decreasing in size (as a percentage) at a rate of deceleration that is today not measurably different from zero. Today's rate of expansion would appear to be constant, both relatively (as a percentage) and absolutely (measured in Planck units). Indeed it has always been constant in absolute terms.

The model is thus consistent with a period of inflation—if by "inflation" we mean a period of time in which the rate of expansion, on a percentage basis, was much larger than it is today. But it does not require the postulation of a *special* inflationary mechanism.

Nothing can circumnavigate the expanding universe

Note that this hyperspherical model postulates a universe that is finite, unbounded, and growing, with a curvature that once was large, but has now effectively reached an asymptotic value of zero. Light traveling on the surface of this expanding hypersphere can continue in a straight line forever, but would never return to its starting point. If we were somehow to "freeze" the universe's expansion, a wave of light would be able to circumnavigate the hypersphere in roughly the age of the universe times 2π. But the distance light would need to travel in an unfrozen, expanding universe would grow at a rate faster than light could travel along it.

The calculation is so simple that it barely needs to be performed. If the radius (age) of the universe at some moment is R (in Planck units of time), the circumference of a hypersphere of radius R is $2\pi R$ (in Planck units of distance). After an additional number of Planck layers have accumulated over a time increment r, the new circumference is $2\pi (R + r)$, an increase of $2\pi r$ Planck distances. But light traveling along the circumference would, during that time period, only be able to move r Planck units, which is a number quite a bit smaller than $2\pi r$.

There are large regions of the universe that never can, and never could, communicate with each other

The conclusion that light cannot complete a circumnavigation in such a universe is also clear when one considers the geometry of light cones in an expanding hyperspherical universe. This seems so obvious that I hesitate to include an illustration of it. Nevertheless, I have done so (Figure 1).

In a flat universe, any observer's light cone (X) will eventually intersect anyone else's light cone (Y), as in Figure 1A. But even if every local area in the universe has a very flat curvature, this does not necessarily imply that the universe as a whole is flat.

Consider the planar cross-section diagrammed in Figure 1B. If you are on the uppermost point of this expanding hypersphere, your light cone delimits two 45-degree angle regions off of the vertical (in the illustration's arrangement) into your future (which in Figure 1B is termed "U"). Your opposite standing on the lowermost point has a light cone into the future that is 180° in the opposite direction (headed "D"). Those light cones never intersect. Likewise, an observer at "R" in Figure 1B would experience a future light cone expanding to the right, and one on the leftmost point "L" would have a future expanding to the left. All of these light cones are non-overlapping, a fact that is another proof of the impossibility of circumnavigating the universe's circumference.

If we were to have the ability to go back to the universe when it was in its earliest, smallest stages, we would see that the currently observable section of our universe would have originated in a narrow region of a tiny, seething hypersphere. Some sort of near energy equilibrium would be reached within the boundaries of that region, without the necessity of having equilibrium established throughout the entire surface of the hypersphere. In the language of Figure 1B, the "U" region might be very different from the "D", "R", and "L" regions. Or it might not be very different. Neither of these facts are ones that future residents of those subregions would ever be able to communicate to each other, or to observe. In a universe as old as ours is now, the curvature everyone measures would be extremely close to zero—as far as our telescopes can see. We will never be able to *observe* the fact that the universe as a whole is still nearly spherical.

The universe cannot contract back to a singularity

It follows that the universe cannot contract back into a singularity. Matter exists in various parts of the universe that cannot communicate with each other. There is no way in which all of the universe's matter can come back together at one location. At most, one could imagine the end state of the universe as a number of black holes, each having accumulated all the mass in its own region of space (followed by their eventual evaporation). But those black holes cannot move fast enough to encounter the black holes that must be presumed to exist on the various poles of the universe.

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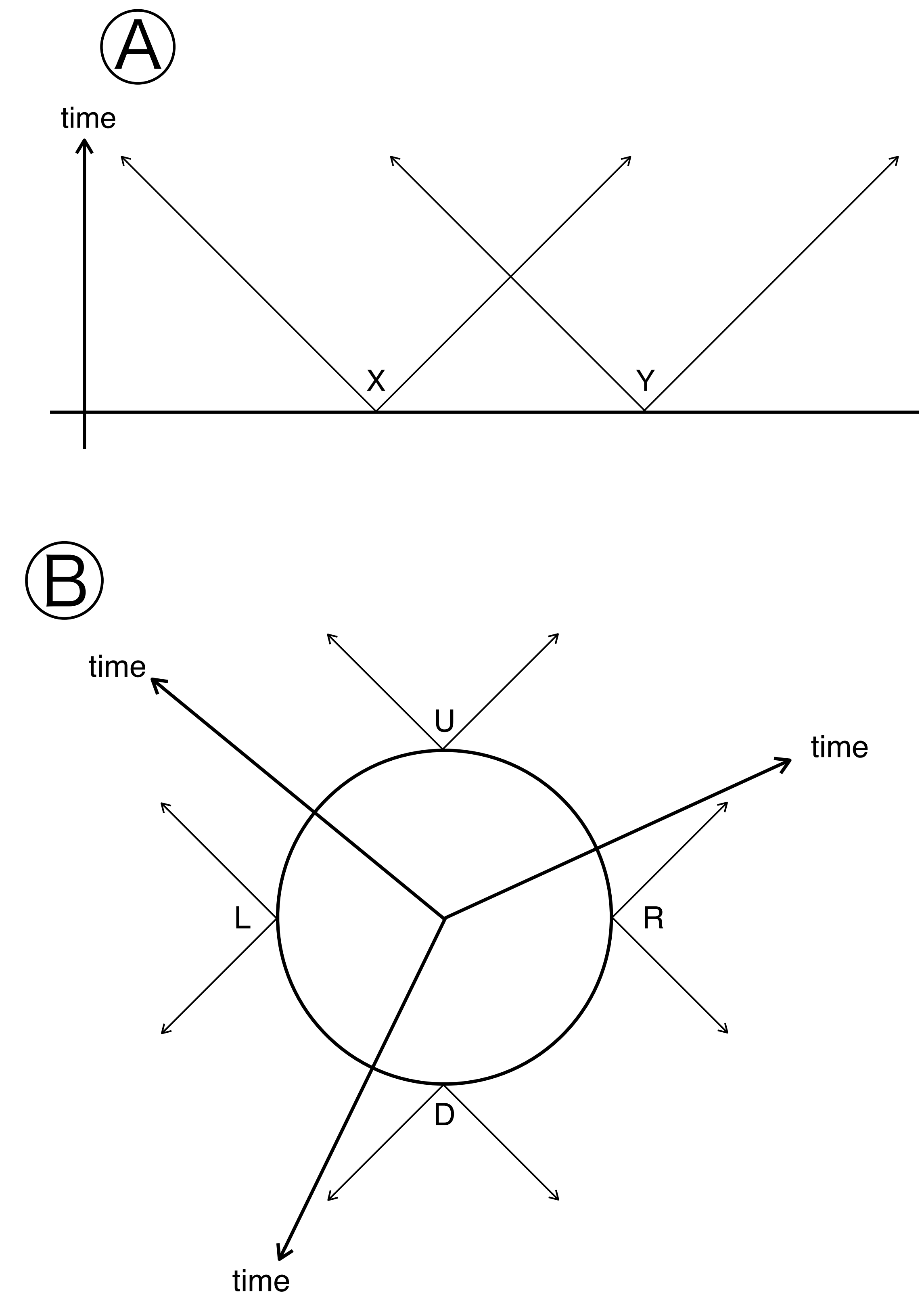
Figure Captions

Figure 1

A:
In a universe with a flat geometry, it is merely a matter of time before the light cones from two different points in space intersect. Thus, any part of the universe can eventually be observed from any specific point (X or Y) in the universe.

B:
In a universe that has positive curvature overall, light cones into the future from two different locations might never intersect (viz., the light cone for U will never intersect those for D, L, and R). Thus, there must exist parts of the universe that can never be observed from one's particular vantage point.

Note: For simplicity, this illustration fails to take account of how, in an expanding universe, light cones will form curves when graphed from the perspective of spacetime (Harrison, 2000, Figure 21.10). This does not alter the fundamental insight, however.



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